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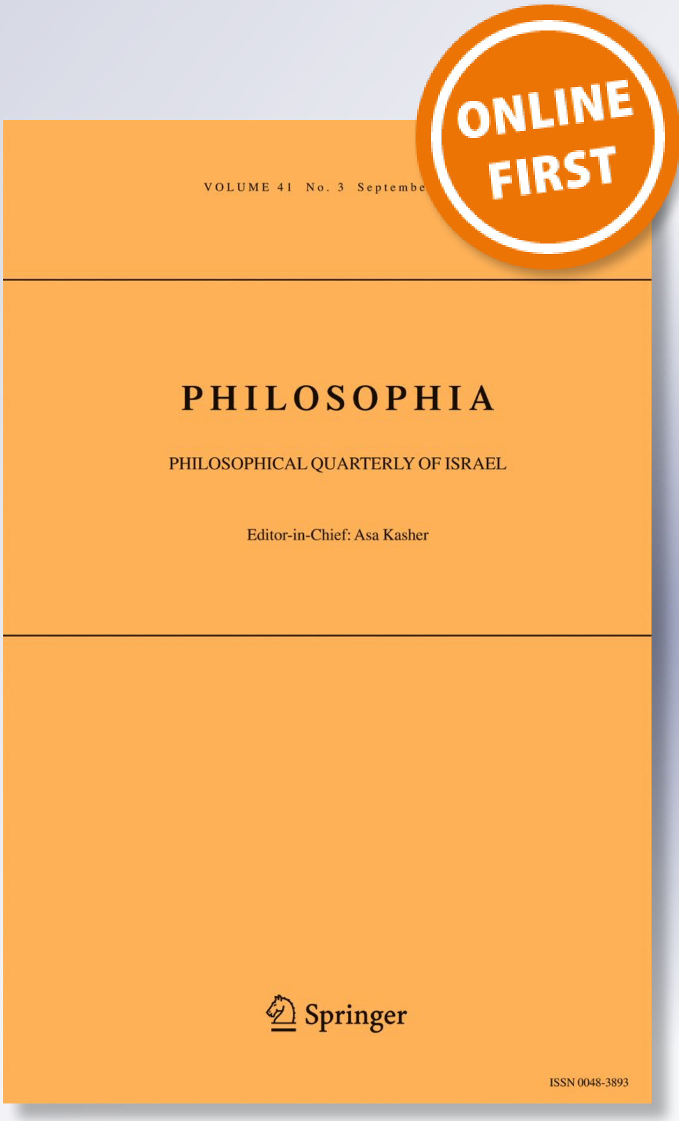

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What Does God Know but can't Say? Leibniz on Infinity, Fictitious Infinitesimals and a Possible Solution of the Labyrinth of Freedom

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Abstract
Despite his commitment to freedom, Leibniz' philosophy is also founded on pre-established harmony. Understanding the life of the individual as a spiritual automaton led Leibniz to refer to the puzzle of the way out of determinism as the Labyrinth of Freedom. Leibniz claimed that infinite complexity is the reason why it is impossible to prove a contingent truth (in a finite number of steps). But by means of Leibniz' calculus, it actually can be shown in a finite number of steps how to calculate a summation of infinite parts. It appears that the analogy Leibniz drew between the mathematics of infinite series and the logic of contingent truths did more harm than good. A solution consistent with Leibniz' perception of infinity is proposed. Alongside the existence of the aforementioned analogy, it is based on a disanalogy between the mathematics of infinite series and the logic of infinitely complex truths. This is a disanalogy that Leibniz had already used to solve the Labyrinth of Continuum which he declared more than once could, like the Labyrinth of Freedom, be solved by means of the nature of infinity. The solution of both labyrinths is based on the fictitiousness of the infinitesimal.

Keywords Leibniz · Infinity · Labyrinth of freedom · Labyrinth of continuum · Fictitious infinitesimal

The complex problem of free will has bothered philosophers throughout the generations. Leibniz also gave it a great deal of thought because his analytical approach to truth and his idea of pre-established harmony constituted a major threat to the existence of freedom. Leibniz made every effort not to arrive at

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the deterministic conclusions. The solution to the problem, which Leibniz called the Labyrinth of Freedom, was derived, to his surprise, "from mathematical considerations on the nature of infinity."[1] This article will attempt to: (a) present the Labyrinth of Freedom against the background of Leibniz' philosophy of analytical truth and especially in the context of the development of his infinitesimal calculus; and (b) propose a solution to the Labyrinth of Freedom based on Leibniz' unique treatment of the concept of infinity. Sections 1–3 are devoted to the first objective and Sections 4–8 to the second. At the same time, I will try to explain why Leibniz was so surprised that the solution to the Labyrinth of Freedom lay in the infinite.

According to my suggestion, the solution of the labyrinth of freedom is similar to Leibniz' solution of the labyrinth of continuum, meaning that it is based on the lack of similarity between mathematics and the field of the labyrinth (logic in the case of freedom and physics in the case of continuum). For this reason, in this article I walk heavily on the path paved by Richard Arthur: I fully embrace Arthur's interpretation of Leibniz' perception of infinity (following Hidé Ishiguro) and also the way in which Arthur clarifies Leibniz' solution of the labyrinth of continuum. But in the case of the labyrinth of freedom, it seems that Arthur is not taking the necessary step from his own research. Like many other commentators, Arthur sees the solution of the labyrinth of freedom as based on basic mathematical knowledge related to incommensurables rather than on infinitesimal calculus. However, as we shall see below, Leibniz' infinitesimal calculus actually created a serious problem for him in solving the labyrinth of freedom. Therefore, the solution seems to lie in identifying the difference between what infinitesimal calculus allows mathematically and what can be demonstrated logically. This difference is due to the fictitious of the Infinitesimal, and it was crucial for Leibniz, As Arthur shows so convincingly, in solving the labyrinth of continuum.

1 Introduction: The Labyrinth of Freedom and the Analytics of Truth

Already at the start of his career, Leibniz took an interest in the organization of knowledge. His universal language project is related to his efforts to provide a definition of factual truth based on logical and mathematical precision. However, the abstract structure of logic and mathematics is either too narrow or too broad to express contingent truth. On the one hand, the inferential laws of logical deduction are derived, in Leibniz' opinion, from the principle of contradiction, which is an absolute truth. The abstract information that they provide (such as: if AB, then A) is true in any possible world since its denial contains a contradiction, and therefore it is too narrow to express the truth of a particular contingent world. On the other hand, logic makes it possible to formulate arguments concerning objects that do not exist in our actual world, since the space of possibilities that do not include a contradiction from the logical point of view is much broader than that which is suited to a particular contingent world. Therefore, the investigation of necessary truths, as well as

[1] 1689?, *On Freedom*, FC 179, AG 95.

consistent possibilities, is simpler than the investigation of contingent truths. The former does not take into account the particular actual world that was chosen to be realized and deal with the validity of the rational process only. In contrast, the investigation of contingent truths is focused on content rather than the form of the rational process, since it examines the unique content of the actual world. Leibniz put this succinctly: necessary truths and the space of possibilities are governed only by the principle of contradiction. In contrast, contingent truths are also governed by the principle of sufficient reason.[2]

A further comment on the relation between form and content in Leibniz' investigation of truth: If knowledge of the world included everything that logic permits, then Hobbes' assumption would be verified. In other words, to define would mean to combine without any rules and to connect without any restrictions different concepts one to the other. The truth is represented by means of signs and logical operators but in Hobbes' opinion these signs are derived from an arbitrary choice of language and therefore the mind cannot arrive at the truth but only at a convention.[3] Leibniz disagreed with Hobbes' viewpoint and felt that the use of arbitrary signs or primary definitions does not make the truth itself arbitrary. According to Leibniz, despite the arbitrariness of signs, there is a fixed pattern between them that obeys the true nature of the signified things, similar to a fixed ratio that is preserved between infinitesimal quantities in his calculus. This order or regularity is what characterizes the truth as analytical, namely, it contains within it the confirmation of its validity. Leibniz stated, without claiming the ability to describe the nature of this order, that unlike mere possibility, "all true predication has some basis in the nature of things."[4] Any true proposition is based on "some" connection between the subject and the predicate.[5]

Leibniz' predicate-in-notion principle, which refers to the statement 'S is P', implies that the notion of the predicate P is contained in the notion of the subject S. This concept-containment principle in which the predicate is in the subject (*praedicatum insest subject*) characterizes the innate order of truth, and is the reason that "a definite analogy between characters and things" is "the basis of truth", and that contingent

[2] 1686, *Discourse on Metaphysics* §13; 1714, *The Monadology* §46, §§54–55; 12 Feb. 1716, *Leibniz to Clark*, 5th letter §9, G VII 390, L 697). Indeed, Leibniz' basic principle that "nothing is without a reason" (1686?, *Primary Truths*, C 519, AG 31) is related to the analytics of truth – and as such it is relevant to both the necessary and contingent truth (Couturat 1994 (1902), 3; Parkinson 1995, 207–208). But it appears that without the contingent domain, it is doubtful whether Leibniz would have bothered mentioning the principle of sufficient reason in relation to necessary truth.

[3] Hobbes, *De Corpore* (1966, vol. 1), Book I, ch. 3.

[4] 1686, *Discourse on Metaphysics* §8, G IV 433, AG 41; August 1677, *Dialogue*, G VII 193, L 185.

[5] "[A]lways, in every true affirmative proposition, necessary or contingent, universal or particular, the concept of the predicate is in a sense included in that of the subject [*praedicatum insest subjecto*]; or else I do not know what truth is. Now, I do not ask for more of a connection here than that which exists objectively [*a parte rei*] between the terms of a true proposition [...] since *there must always be some basis for the connection between the terms of a proposition, and it is to be found in their concepts*." (4/14 July 1686, *Leibniz to Arnauld*, letter X, G II 56, LA 63-64). As a result, Bertrand Russell and Louis Couturat claimed early in the twentieth century that Leibniz' overall philosophy should be viewed on the basis of his logic, i.e., the logical nature of a true proposition (Couturat 1994 (1902), 2–8; Russell 1900, 40–53). Many have criticized Russell and Couturat's sweeping claim, but no one would disagree that Leibniz never changed his mind about the the crucial importance of his predicate-in-notion principle.

truths, and not only necessary truths, "consist not in the arbitrary element in their characters but in the permanent element in them, namely, in their relation to things."[6]

For Leibniz, this was not just a matter of logic, but primarily involved metaphysics, which created a problem for him. The process of combination that generates complex concepts exists in the mind of God, and gives complex concepts their validity as possibilities, regardless of the question of whether they are chosen in the end to be realized in the actual world.[7] As claimed by Nachtomy, God creates possibilities in his mind by means of thinking about combinations of his attributes.[8] Since the attributes of God are absolutely simple and positive (i.e. they do not include negation), there is no possibility of describing their essence by means of analysis. According to Leibniz, the limitation of his logical analysis due to the inaccessibility of elementary concepts, is not meant to compromise analytical ability.[9] Proving the truth of arguments is in fact accessible to man because in order to prove an argument there is no need to break it down into concepts that are not given to further analysis; it is enough to describe an identity argument. Thus, for example, in order to prove that "all men are rational", one needs to break down the complex concept "man" into the components "animal" and "rational". The claim that "all rational animals are rational" is an identity argument whose truth is proved logically without breaking down the concepts any further.[10] Therefore, according to Leibniz, a true proposition is one that can be used to demonstrate that the predicate is included in the subject by means of reducing it to identity arguments or arguments from which it can be inferred logically (by means of the rules of inference) that the argument is true. Thus, Leibniz preserved the pretension of organizing knowledge according to a combinatorial order and by means of a symbolic alphabet, where the initial elements are not necessarily absolutely simple but rather "primary for us."[11]

We now turn to the labyrinth. During the 1680s, Leibniz realized that the definition of truth as analytical, that is, based on the predicate-in-notion principle, transforms any contingent truth into a necessary one.[12] It turns out that contingent truths are located in the tension between deterministic necessity and meaningless arbitrariness. Defining contingent truth analytically makes it necessary; defining it synthetically makes it

[6] August 1677, *Dialogue*, G VII 192-193, L 184-185. The relation that Leibniz is referring to in understanding truth is not the extensional relation with which one can sort concepts into groups and types but rather an intensional relation by means of which one can assemble concepts. According to Leibniz, he preferred the intensional approach since he sought to deal with concepts in isolation from the existence of things (April 1679, *Elements of a Calculus*, C 53, PLP 20). The intensional approach, which deals with concepts without regard to the number of their practical examples, is consistent with Leibniz' demand that the proof of the possibility of a concept will serve as a necessary (though not sufficient) condition for its actual realization, i.e. its existence. From a more general perspective, one can see the reason for preferring the intensional viewpoint of Leibniz as related to his logical foundation of the metaphysical harmony between infinitely many things in the world and his perception of rationality as completely analytical (Yakira 1988, 31).

[7] 4/14 July 1686, *Leibniz to Arnauld*, letter X, G II 55, LA 62; 1714, *Monadology*, §43.

[8] Nachtomy 2007, 25.

[9] 1679–1686, *An Introduction to a Secret Encyclopaedia*, C 513-514, trans. by Parkinson 1966, xxvii-xxviii, MP 7-8; cf. 1682–1684, *On the Elements of Natural Science*, LH XXXVII, vi, 3, L 283; Nov. 1684, *Meditations on Knowledge, Truth, and Ideas*, G IV 425, AG 26.

[10] 1679–1686, *An Introduction to a Secret Encyclopedia*, C 518, trans. by Parkinson 1966, xxix.

[11] 1686, *General Inquiries about the Analysis of Concepts and Truths*, C 358, PLP 49.

[12] 1689? *On Freedom*, FC 179, AG 95.

arbitrary. Leibniz was determined to define contingent truths as analytical as well since such a definition creates an internal relation between signs in language similar to the relation between the objects themselves and therefore it can overcome Hobbes' threat of arbitrariness; however, in this way the problem is shifted to the pole of necessity. In other words, Leibniz had to solve the problem in which, on the one hand, the subject acts according to the analytical regularity inherent in its complete concept, but on the other hand it still can choose freely without contradicting the predicates and events already included in its concept.

2 Leibniz' Solution: Infinite Analysis of Contingent Truth

After pondering the issue for many years, Leibniz discovered that he could dismantle the triangular relation between the necessity of an argument, the fact that it a priori includes its truth (that is, the fact that it is analytic) and the fact that its truth can be proven. Leibniz argued that any necessary argument is analytical and therefore can also be demonstrated. However, the opposite is false: not every analytical argument is necessary and therefore not every analytical argument can be demonstrated. In other words, we thought that provability is a trait of analyzability but it appears to be a trait of necessity only. Contingent analytical truth is not provable.[13] Leibniz used this small fissure between the analytical nature of contingent truth and its non-provability to argue that a priori knowledge of a contingent matter only "inclines without necessitating". Therefore, Leibniz declared that he had found the solution to the Labyrinth of Freedom:

> [H]ere lies hidden a wonderful secret, a secret that contain the nature of contingency, that is, the essential difference between necessary and contingent truths, a secret that eliminates the difficulty concerning the fatal necessity of even those things that are free.[14]

So what is that secret? Several years later, Leibniz revealed that he was surprised to discover that the secret had to do with the infinite:

[13] Leibniz identified infinite analysis as an expression of non-provability of contingent truth. However, Blumenfeld argues that the inability to complete the proof (due to its infinite length) is not a reasonable solution to the contingency problem. Assuming that the analytical cannot necessarily be demonstrated, it is possible to come to one of the following conclusions: One can conclude—as Leibniz did—that there is contingent analytics (which cannot be demonstrated), but it is absolutely possible to simply think that there is a necessary truth that cannot be demonstrated. Leibniz did not justify why he began from the assumption that the necessary can be demonstrated with certainty and therefore the analytical is the source for the contingent, instead of presenting a much more reasonable argument: that not all necessary truths can be demonstrated (Blumenfeld 1985, 499–500). To this I would respond that Blumenfeld's proposition sounds reasonable only after the famous proof of Gödel that necessary truths in mathematics are not provable (or that an infinite number of steps is required to complete such a proof). It seems to me that there is no point in expecting that Leibniz would hold such a position when at the beginning of the twentieth century mathematicians such as Hilbert still believed that one could prove any true argument in mathematics with a finite number of steps. Even if the alternative chosen by Leibniz is equivalent to that which he rejected, we must refrain from anachronistically viewing the dilemma which he faced (see Lison 2006–7).

[14] 1686?, *Primary truths*, C 518-519, AG 31.

> At last a certain new and unexpected light shined from where I least expected it, namely, from mathematical considerations on the nature of infinity.[15]

Analysis of complex concepts down to more basic ones (which as mentioned are not necessarily simple) is meant to end in an identity argument that demonstrates their truth. This basic idea, which relates to Leibniz' a priori predicate-in-notion principle, is meant to be reflected also in complex contingent truths, namely also in the concept of a specific individual (such as the "first man") and not only in the analysis of the general term "man" by means of the concepts "animal" and "rational". In Leibniz' opinion, the important difference between necessary truths and contingent truths is that the latter cannot be proven and this is to do with infinity. Just as an incommensurable ratio of numbers infinitely approaches a given number, but never reaches it, so also the containment of the predicate in a contingent subject infinitely approaches the identity argument that constitutes the boundary of the analysis, but such an identity argument always remains out of reach.[16] It appears that Leibniz believed that the analogy between mathematical investigation of quantities and the logical investigation of truths about the world is the key to the solution of the Labyrinth of Freedom:

> [T]he relation of contingent to necessary truths is somewhat like the relation of surd ratios (namely, the ratios of incommensurable numbers) to the expressible ratios of commensurable numbers. […] And so I think I have disentangled a secret which had me perplexed for a long time; for I did not understand how a predicate could be in a subject, and yet the proposition would not be a necessary one. But the knowledge of geometry and the analysis of the infinite lit this light in me, so that I might understood that notions too can be resolved to infinity.[17]

let's turn for a moment to Leibniz' view of infinity. In his writing on the mathematics and geometry of infinity, Leibniz had in mind the syncategorematic view of infinity, according to which an infinite number is not possible since for every number x there is always a number y that is larger. This is in contrast to the categorematic view of infinity, according to which an infinite number does in fact exist since there is a y greater than any other number x, a view that Leibniz rejected.[18] In this context, there is also a clear relation between large infinity and

[15] 1689?, *On Freedom*, FC 179, AG 95. Starting from the second half of the 1680s, and until the end of his career, Leibniz was consistent in the use of infinity to define contingent truths. See, for example, 5 August 1715, *Leibniz to Bourguet*, G III 581, L 664.

[16] 1679–1686?, *A Specimen of the Universal Calculus*, A 6.4 280-289, G VII 200, W 98-99.

[17] 1686, *Necessary and Contingent Truths*, C 17-18, MP 97.

[18] "It is perfectly correct to say that there is an infinity of things, i.e. that there are always more of them than one can specify. But it is easy to demonstrate that there is no infinite number, nor any infinite line or other infinite quantity, if these are taken to be genuine wholes. The Scholastics were taking that view, or should have been doing so, when they allowed a 'syncategorematic' infinite, as they called it, but not a 'categorematic' one. The true infinite, strictly speaking, is only in the *absolute*, which precedes all composition and is not formed by the addition of parts." (1709, *New Essays*, book II, ch. xxvii [of infinity], NE 157. Also: 2 Feb. 1702, *Leibniz to Varignon*, GM IV 93, L 543; 1 Sep. 1706, *Leibniz to Des Bosses*, G II 314, LR 53. For a logical formulation of the argument, see: Arthur 2001b, 107; Levey 2008, 109; Arthur 2015, 145–146).

small infinity—in an infinite series, the last number is in the infinite place. If an infinite number lacks meaning, then the infiniteth member of an infinite series is impossible.[19] If the last term is always missing, then an infinite quantity in a converging series cannot produce a whole and the distance between the series and its limit can never disappear. Therefore, the justification for the reliability of the calculation that allows for the summation of infinite series lies in ignoring this infinitesimal gap. However, this is problematic since in a converging infinite series of discrete terms, the gap between the series and its limit always remains, even if it is continually narrowing. Indeed, the arithmetic processing of an infinite series, which is focused on the terms and the differences between them, was indeed very fruitful for Leibniz in identifying the inverse relation between summation and differentiation (as we shall see below), but it also raised a major problem in the justification of the calculus. In contrast, a geometric treatment of an infinite series is not focused on the real size of the differential but on the progress of convergence itself toward the limit of the series – a limit which appears to be **inside** the series but is actually **outside** it.[20] Geometry provides a better perspective on the continuum and makes it possible to obscure the fact that the mathematical continuum as a whole is created from discrete elements that can be divided further to infinity. That is the reason why in summing up his activity in mathematics, Leibniz pointed to the geometric perspective as more

[19] However, there is no perfect symmetry between large infinity and small infinity. Only large infinity leads to a problem of the whole being equal to its parts (the infinite number is the number of all numbers and therefore constitutes a part of itself since it is included, along with all the other numbers, within the number of all numbers, which is identical to it). Such a contradiction does not exist in the definition of an infinitesimal number and this is perhaps the reason why Leibniz does not prove that the infinitesimal is not possible. As shown by Bassler, Leibniz indeed provided a proof that the infinitesimal is not possible but never came back to it. This proof is based on the fact that an infinite size cannot be limited (Bassler 2008, 146–148). Therefore, it does not relate directly to the infinitesimal as an impossible size. Levey claims that Leibniz never proved that the infinitesimal is not possible even though he declared the existence of such a proof in his correspondence with Bernoulli ("As concerns infinitesimal terms, it seems to me not only that we cannot penetrate to them but that there are none in nature, that is, that they are not possible. Otherwise, as I have already said, I admit that if I could concede their possibility, I should concede their being." 18 Nov. 1698, *Leibniz to Bernoulli*, GM III 551, L 511). Although Leibniz already proved at the end of 1672 that a minimal size is not possible (by proving that a line cannot be composed of undivided points), in that same correspondence with Bernoulli he also emphasized that ruling out a minimum is not the same as ruling out an infinitesimal: "I admit that the impossibility of our infinitesimals does not follow directly from this [...] since a minimum is not the same thing as an infinitesimal"(1699, *Leibniz to Bernoulli*, GM III 535-536, Levey 1998, 56 n7).

[20] "Although it is not at all rigorously true that rest is a kind of motion or that equality is a kind of inequality, any more than it is true that a circle is a kind of a regular polygon, it can be said, nevertheless, that rest, equality, and the circle terminate the motions, the inequalities, and the regular polygons which arrive at them by a continuous change and vanish in them. And although these terminations are **excluded**, that is, are not included in any rigorous sense in the variables which they limit, they nevertheless have the same proportions **as if they were included** in the series, in accordance with the language of the infinities and infinitesimals, which takes the circle, for example, as a regular polygon with an infinite number of sides. Otherwise, the law of continuity would be violated, namely, that since we can move polygons to a circle by a continuous change and without making a leap, it is also necessary not to make a leap in passing from the properties of polygons to those of a circle." (Jan. 1701, *Justification of the Infinitesimal Calculus by that of Ordinary Algebra*, GM IV 106, L 546 (my bold); cf. 1701?, Cum Prodiisset, Child 148-149). See Ishiguro 1990, 79–100.

fruitful and persuasive than the arithmetic perspective in justifying his new calculus.[21]

We now return to the Labyrinth of Freedom. In order to solve the Labyrinth of Freedom, Leibniz provided an analogy between mathematics and logic which is based on the infinite analysis of irrational quantities and is meant to show that a logical analysis also has an 'arithmetic' aspect that is related to the endless process and a 'geometric' aspect that views it as a whole.[22] Leibniz' concluded from this analogy that contingent truth cannot be proven in exactly the same way that an irrational fraction cannot be calculated arithmetically. Only God can perceive contingent truth in its entirety, by means of intuitive knowledge, which provides direct contact with the whole. Therefore, even God, who perceives contingent truth 'geometrically', cannot 'arithmetically' break down the infinite logical analysis into discrete arguments which eventually provide the desired proof. God's perfect a priori knowledge is a result of the analytic nature of contingent truth, but even he cannot complete an infinite analysis of such a truth.[23]

3 Problems Arise: The Analogy Is Too Strong

However, this solution is far from satisfying. According to Di Bella, it appears to be based solely on the approach of the Greek geometricians to the study of irrational numbers, such as could have been formulated even if the infinitesimal calculus had not been created at all.[24] In fact, not only does the solution ignore the calculus developed by Leibniz himself, it appears that the calculus actually turns the situation on its head. It makes the analogy from mathematics to logic too strong. This is because Leibniz did in

[21] "The differential calculus could be employed with diagrams in an even more wonderfully simpler manner than it was with numbers, because with diagrams the differences were not comparable with the things which differed; and as often as they were connected together by addition or subtraction, being incomparable with one another, the less vanished in comparison with the greater; and thus irrationals could be differentiate no less easily than surds, and also, by the aid of logarithms, so could exponents. Moreover, the infinitely small lines occurring in the diagrams were nothing else but the momentous differences of the variable lines." (1714, *Historia et Origo Calculi Differentialis*, Child 53-54).

[22] According to Leibniz, "The doctrine of irrational numbers, like what is contained in book X of the *Elements* [of Euclid]," is compared to "knowledge by intuition"; whereas "common arithmetic" is equalized to "knowledge of simple understanding". Thus "it is impossible to give [a] demonstration of contingent truths" because "irrational proportions [cannot] be understood arithmetically, that is, they cannot be explained through the repetition of a measure." (1685–1689?, *The Source of Contingent Truths*, C 1-3, AG 98-100)

[23] This can be illustrated by means of the following table:

	Commensurable quantities	Incommensurable quantities	Necessary truth	Contingent truth
Analytical a priori	✓	✓	✓	✓
Provable	✓	✗	✓	✗

[24] "…every series, complex and irregular as it may be, and even including miracles, can be reduced to a unitary rule. To claim this, Leibniz plausibly had in mind his mathematical experience in the study of functions. In his texts on infinite analysis, however, the analogy in not usually drawn from some new discovery in infinitesimal calculus, but from the polarity between rational and irrational numbers, already known from antiquity." (Di Bella 2005, 356). Leibniz himself relates explicitly to Euclid's Elements: the shift of the Greeks from arithmetic to geometry allowed them to consider irrational sizes such as π, without relating to the fact that they are incommensurable quantities.

fact prove, in a finite number of steps, the summation of a convergent infinite series of numbers.[25] In other words, the distinction between commensurable and incommensurable quantities is not helpful since the calculus developed by Leibniz is able to overcome the arithmetic constraint and complete the calculation, as we will see immediately. In fact, this may be the reason why Leibniz claimed that he succeeded in solving the labyrinth of freedom only more than ten years after his mathematical discovery of Infinitesimal calculus. He had to find out why, despite the calculus, it was impossible to prove contingency. Essentially, it appears that the effort to solve the Labyrinth of Freedom based on a mathematical analysis of infinite series can lead to a nullification of the important distinction between necessity and contingency. It seems that the analogy between mathematics and logic did much more harm than good, because it may lead to the fact that even contingent truth is ultimately necessary.[26]

Richard Arthur, a commentator known for his contribution to understanding Leibniz' mathematics, probably noticed that the solution to the labyrinth of freedom cannot rely only on the resemblance between contingency and incommensurability, i.e. on mathematics known by the Greeks already. Therefore, he binds the solution with the unbridgeable gap between an infinite series and its limit. In spite of that, in the end, the reason demonstration of contingent truth is not possible according to Arthur is not related to new information based on the calculus, but to infinite complexity only.[27]

If we wish to take this analogy seriously, we must view the identity argument to which the infinite analysis of a contingent truth is meant to converge in the same way that we view the limit of an infinite series. When one analyzes a contingent subject with the goal of proving that some predicate is included within it, the analysis can never be completed and therefore the required identity argument must be **outside** and beyond analysis. However, since a contingent truth is analytic and a priori includes all of its predicates, it appears that this identity argument must be included **within** the analysis and therefore in the end is attainable. This vagueness as to the status of the last identity argument led a number of commentators to claim that the absence of

[25] "Leibniz thinks for example that we must understand clearly that we can rigorously prove that 1/2 + 1/4+ 1/8 + … = 1, although the number of terms of the series is not finite. But to give a proof of a truth involving infinitely many terms is surely not to give a proof which itself has 'infinitely' many steps." (Ishiguro 1990, 194)

[26] The problem can be demonstrated by means of the table presented above:

	Commensurable quantities	Incommensurable quantities	Necessary truth	Contingent truth
Analytical a priori	✓	✓	✓	✓
Provable	✓	✓	✓	✓

[27] "Terminating [Leibniz'] series after some very large finite number of terms gives us a good value for $\pi/4$, but even then there will be 'a new remainder that results in a new quotient', which itself is explicable as a series of infinitely many further quotients. This, Leibniz held, is analogous to a contingent truth: a contingent truth 'involves infinitely many reasons, but in such a way that there is always something that remains for which we must again give some reason' (*On the Origin of Contingent Truths*; A 6.4 1662, AG 99). But given Leibniz's doctrine explain above that all demonstrations must involve only a finite number of steps, this means it is impossible for there to be a demonstration of a contingent truth, since such a demonstration would require an infinite analysis." (Arthur 2014, 95–96)

an identity argument from the analysis is due solely to an epistemic limitation. In the end, the identity argument is meant to exist somewhere at the end of the analysis that **we** do not manage to complete. If God knows the contingent truth in full, then the analysis to infinity creates contingency and freedom only in appearance and in the end everything is necessary.[28]

It is important to note at this point that if the solution to the labyrinth of freedom is based on infinite complexity, it reaches a similar problematic conclusion. This is because infinite complexity is not accessible to our finite minds, but it is spread and open in God's mind. Leibniz did hint at the connection between infinity and solving the labyrinth, but that does not mean that the key to the riddle is simply infinite complexity, but something else connected to infinity, otherwise this solution leads to freedom only for appearance.[29]

On the other hand, there are commentators who claim that an infinite analysis of a contingent subject cannot end in an identity argument, even if God carries out the analysis. Proof is attained when the analysis arrives at an identity argument and proof by negation is achieved when the analysis arrives at a contradiction to an identity argument. Leibniz defined contingency as a possible argument, namely an argument whose negation does not lead to a contradiction. In other words, the analysis of a contingent argument cannot be decided by affirmation or by negation by means of an identity argument.[30] If this is the case, then the identity argument is in any case beyond the reach of God. According to the analogy constructed by Leibniz, divine intuition is able to perceive contingent truth in full, despite the unattainability of the identity argument that proves it. God can see whatever is in an infinite series that is included in a contingent subject, but is unable to see the end of the analysis since such an end simply does not exist.[31] But if God perceives the truth in its entirety—and this is an analytical truth that a priori includes the identity argument that proves it—why does the identity argument remain concealed even from him?

The solution of the Labyrinth of Freedom is meant to explain why even God is unable to prove that a particular event is included in the complete concept of a given individual. Therefore, it appears that the solution must be based not only on an analogy but also on a disanalogy between the mathematics of infinite series and the logic of infinitely complex truths.[32] This is a disanalogy by means of which Leibniz already

[28] Russell 1972 (1903), 378 n. 8; Lovejoy 1936, 174–175; Curley 1972, 80.

[29] More on the infinite complexity and the problems in identifying infinite complexity as the source of the solution, in section 6 below.

[30] Rescher 1967, 46; Adams 1982, 258. According to Sleigh, there is a difference between an infinite analysis and an analysis that has no end (Sleigh 1982, 227–236). This difference in his opinion can be used to explain how God knows contingent truth: God can complete an infinite analysis. The problem is that this interpretation is not consistent with a syncategorematic view of infinity which Leibniz held.

[31] "But in contingent truths [...] the resolution proceeds to infinity, God alone seeing, not the end of the resolution, of course, which does not exist, but the connection of the terms or the containment of the predicate in the subject, since he sees whatever is in the series." (1689?, *On Freedom*, FC 181-182, AG 96)

[32] The solution can be illustrated using the above table:

	Commensurable quantities	Incommensurable quantities	Necessary truth	Contingent truth
Analytical a priori	✓	✓	✓	✓
Provable	✓	✓	✓	✗

solved a different labyrinth—one no less complex—that is related to the nature of the physical continuum. Indeed, Leibniz declared more than once that the two labyrinths—the Labyrinth of Continuum and the Labyrinth of Freedom—can be solved by means of "mathematical considerations related to the nature of infinity." Leibniz solved the Labyrinth of Continuum already in 1676, during the same period in which he developed the infinitesimal calculus. In order to understand exactly how infinity solves the Labyrinth of Freedom, we need to go through the path Arthur already made in his interpretation, since his approach is likely to be correct in understanding the solution of the labyrinth of continuum based on Leibniz' infinity perception. In fact, we need to extend this path to apply even to the labyrinth of freedom, though Arthur himself has refrained from doing so. Therefore, we must: (a) take into account the fictitious status of the infinitesimal in Leibniz' calculus; (b) understand the solution that Leibniz proposed to the Labyrinth of Continuum; (c) become familiar with the power of the relationship between the two labyrinths; and finally (d) determine whether the solution of the Labyrinth of Continuum can be modified to solve the Labyrinth of Freedom. I believe that it can.

4 The Key to Leibniz' Solution: Infinity and the Infinitesimal as Fictitious Quantities

Leibniz' infinitesimal calculus enables the summation of infinitely descending quantities. Leibniz believed that in order to justify such a calculus mathematically, we need to justify ignoring infinitely small quantities due to their negligible size, even though they do not disappear completely. Ignoring mathematical quantities is not possible as long as they represent real quantities. Therefore, the key to allowing one to ignore the infinitesimal lies in its identification as a quantity that cannot be real, even though it can be possible. Indeed, from Leibniz' perspective, viewing the infinitesimal as fictitious was the basis for justifying his calculus.[33]

Leibniz approached infinite series or quantities from the proportional perspective. As mentioned, the term "infinity" cannot relate to a given infinite quantity (which is absurd according to Leibniz) but rather to a finite quantity that can be increased as much as we want. Such a quantity can be called "infinite" by Leibniz since proportionally it is so much larger than any other quantity that describes a given variable. In the same way, a tiny finite quantity that can be further reduced as much as we want can be called "infinitesimal" since proportionally it is so much smaller than any other quantity that describes

[33] "For my part I confess that there is no way that I know up till now by which even a single quadrature can be perfectly demonstrated without an inference ad absurdum. Indeed, I have reasons for doubting that this would be possible through natural means without assuming fictitious quantities, namely infinite and infinitely small ones." (Fall 1675-Summer 1676, *De Quadratura Arithmetica*, 35, trans. by Arthur 2008, 25 n16).

that same given variable.[34] Leibniz' infinitesimal calculus is based on an "incomparable" ratio that nonetheless is maintained between the various levels of the differentials: the ratio between an infinitely small quantity (dx) and a given quantity (x) is "incomparable", but even so is still similar to the ratio between the given quantity (x) and an infinitely large quantity ($\int x$), which is also, as mentioned, "incomparable".[35] It is not the particular size of the infinitesimal that is important but rather the fixed ratio that it maintains with differentials of higher and lower orders. The focus on the fixed ratio between different levels of differentials rather than the size of a particular differential allowed Leibniz to recognize its fictitious nature. Indeed, the identification of the fixed ratio between a given series of terms, a series of summations of these terms and a series of differences between these terms is what enabled Leibniz, already at the beginning of his mathematical career, to discover the inverse relation between summation and differentiation.[36] For example, a first-order differential series is actually the sum series of the second-order differential series, and the original series can be considered as the sum series of the first-order differential series. In this way—and this is the important point—Leibniz was able to undermine the identification of any series as "original". In exactly the same way, the focus on the fixed ratio between infinite levels of differentials makes irrelevant the question of what the real parts are that make up the continuum.[37]

[34] "Just as I have denied the reality of a ratio, one of whose terms is less than zero, I equally deny that there is properly speaking an infinite number, or an infinitely small number, or any infinite line, or a line infinitely small [...]. The infinite, whether continuous or discrete, is not properly a unity, nor a whole, nor a quantity, and when by analogy we use it in this sense it is a certain *facon de parler*; I should say that when a multiplicity of objects exceeds any number, we nevertheless attribute to them by analogy a number, and we call it infinite. And thus I once established that when we call an error infinitely small, we wish only to say an error less than any given, and thus nothing in reality. and when we compare an ordinary term, an infinite term, and one infinitely infinite, it is exactly as if we to compare, in increasing order, the diameter of a grain of dust, the diameter of the earth, and that of the sphere of the fixed stars." (1712, *Acta Eruditorum*; GM V 389, trans. by Jesseph 2008, 231 n30)

[35] "It would suffice here to explain the infinite through the incomparable, that is, to think of quantities incomparably greater or smaller than ours [...] But at the same time we must not consider that these incomparable magnitudes themselves are not at all fixed or determined but can be taken to be as small as we wish" (2 Feb. 1702, *Leibniz to Varignon*, GM IV 91, L 543). As mentioned, it appears that Leibniz already in 1676 arrived at the conclusion that the infinitesimal can be viewed spatially rather than quantitatively and continued to hold this view from that point onward. (11 Feb. 1676, *On the Secrets of the Sublime*, A 6.3 475, LLC 49; cf. Feb. 1689, *Tentamen De Motuum Coelestium Causis*, GM VI 150-151)

[36] "The consideration of differences and sums in number sequences has given me my first insight, when I realized that differences correspond to tangents and sums to quadratures." (28 May 1697, *Leibniz to Wallis*; GM IV 25, trans. by Bos 1974, 13)

[37] The infinite division of the differential makes possible various level of differentials, which can be ordered by means of a fixed ratio. This fixed ratio of differentials of different orders is important to Leibniz to oppose Cavalieri, Wallis and other mathematicians at that time that believe the differential should be considered as indivisible and the difference between infinite series and its sum should be considered as null. ("For my quantity a is the same as your dx, except that my a is nothing and your dx infinitely small. Then when those things are neglected which I hold should be neglected in order to abbreviate the calculation, that which remains is your minute triangle, which according to you is infinitely small, but according to me is nothing or evanescent", 30 July 1697, *Wallis to Leibniz*, GM IV 37, trans. by Jesseph 1998, 24). However, Leibniz argued, if the differential is divisible, then there is no necessary reason why the differential of the differential will not be divisible either. In Leibniz' view, Cavalieri's method of indivisibles relates to a special situation in which $dx = 1$ and $ddx = 0$; But this limitation necessarily prevents recognizing the existence of differentials of higher orders and limits the generality of calculus (Late 1675 – summer 1676, *De Quadratura Arithmetica* 24, trans. by Arthur 2008, 25). For this reason, Leibniz claimed in an official publication ten years later that the omission of differentials from the equation (or their identification as zero) are appropriate only in the special case in which the rate of the variable's progress is fixed (*Acta Eruditorum*, June 1686, *De Geometria Recondita et Analysi*, GM VI 233, trans. by Bos 1974, 79).

Since it cannot be determined which level of infinitely small parts indeed makes up the real continuum, then none of them actually do. In this way, separation is achieved between mathematics and reality, which paves the way for the solution of Leibniz' Labyrinth of Continuum, as we will see below.[38] In this way, spatial or proportional thinking, rather than arithmetic thinking, forms the foundation of the view of the infinitesimal as fictitious. Despite its importance, the infinitely small figure cannot be identified quantitatively. It is a fundamentally undefined mathematical entity, which even though it is possible cannot be real and therefore must be an imaginary fiction.

5 Solving the First Riddle: The Labyrinth of the Continuum and its Solution by Means of Infinity

We must now move on to look at the function played infinity in general and the fictitious status of the infinitesimal in particular in the solution of the Labyrinth of the Continuum proposed by Leibniz. Infinite divisibility of matter creates confusion as to the continuity of the matter perceived by the mind. Is it an illusion? And if continuity is an illusion, why is matter itself not an illusion by the same token? And if matter does in fact exist but is not continuous, how exactly does change occur in matter? How is the shift from movement to rest made possible? From heating to evaporation? From existence to non-existence? According to Arthur and Levey, Leibniz outlined the solution to the Labyrinth of the Continuum by means of the "model of folds" which he formulated for the first time already in 1676, a few months after developing the

[38] Already at the beginning of 1676, Leibniz raised for the first time the possibility that the Cartesian identity between space and matter is mistaken because space is a whole continuum while matter is discrete and aggregative (11 Feb. 1676, *On the Secrets of the Sublime*; A 6.3 473-474, LLC 47). Leibniz wondered about this since he found it difficult to explain the continuity of matter. Mathematically, the spatial definition of the infinitesimal by means of an "incomparable" ratio points to the fact that this is an undifferentiated quantity which is defined only as a derivative of the whole that preceded it. The idea that the ideal whole is precedent to its parts (in contrast to the priority of the parts to the whole on the real level), leads to the precedence of the summation over the infinite series and makes it possible to ignore the difference between them. However, in this manner, the way out of the Labyrinth of Continuum is founded on an unbridgeable gap between reality and mathematics and infinity is perceived as only potential. Indeed, later in his career, Leibniz claimed that the solution of the Labyrinth of Continuum lies in the gap between mathematics which is based on the whole being precedent to the parts and physics which is based on the simple preceding the complex ("For space is something continuous but ideal, whereas mass is discrete, indeed an actual multitude, or a being by aggregation, but one of infinite unities. In actual things, simples are prior to aggregates; in ideal things, the whole is prior to the part. Neglect of this consideration has produced the labyrinth of the continuum." 6 Sep. 1709, *Leibniz to Des Bosses*, G II 379, LR 245; cf. 19 Jan. 1706, *Leibniz to De Volder*, G II 282, L 539, AG 185). At first glance, this view of infinity as potential is not consistent with Leibniz' view of infinity as syncategorematic and actual. This is because potential division does not have to take into account that a whole can be divided into half or into thirds but never into halves and also into thirds. Potential parts can be the result of all the possible divisions since they are undifferentiated. In contrast, actual parts must be differentiated since they together constitute the actual whole. The conclusion that we cannot show how the actual continuum is constituted and that the infinitesimal parts are fictitious spills over to the view of infinity as only potential and is no different from that of Aristotle. But on second glance and despite the separation between ideal mathematics and actual existence that enables us to describe the infinitesimal as fictitious, Leibniz was careful to emphasize that actual reality nonetheless operates according to the eternal rules of mathematics (1702, *Leibniz' Reply to Bayle's article 'Rorarius'*, G IV 569, L 583). Therefore, Leibniz' position is more complex than simply restricting the infinitesimal to the status of an undifferentiated fiction, and this is manifested in the metaphysical meaning of his Principle of the Continuum (for more on that, see Levey 2008, 128–132).

principles of his infinitesimal calculus.[39] The justification of the calculus by the fictitious status of the infinitesimal implies that continuity characterizes mathematical convergence but does not exist in reality. In reality, one can always continue to divide matter and it is impossible to ignore the last part, to finish the division and put together a whole from infinitely many parts. Indeed, according to Leibniz' model of folds, matter is totally flexible and its division into infinite sub-matters and internal motions will never reach indivisible points since it is based on a syncategorematic view of infinity. The image of the folds is meant on the one hand to prevent the existence of the indivisible last part, that is, the completion of the infinite division, and on the other hand to leave matter as a consistent thing over time.[40]

The model of folds enabled Leibniz to provide matter with a presentation that had a certain stability despite its expected crumbling when being infinitely divided. But the model of folds has additional explanatory power by making it possible to relate to motion as something imaginary.[41] When Leibniz claimed that motion is imaginary, he was not claiming that it is not possible but rather that it is undifferentiated due to the infinitely frequent changes that occur in it. Actually, motion never ends and changes in matter never advance to full completion. Changes occur only in endless continuity without arriving at the final point. Mathematical quantities can be ignored because they are potential and derived from the existence of the ideal whole. But in the actual division of matter, they cannot be ignored and therefore physical processes proceed continuously without a final destination that completes them. Absolute rest and the exact configuration of physical

[39] Levey criticized Leibniz' "model of folds" as incoherent (Levey 1999, 148), whereas Arthur (2001a, lxxvi; Arthur 2014, 84–85) saw it, alone with the important role of force, as successful solution to the Labyrinth of Continuum.

[40] "It is just as if we suppose a tunic to be scored with folds multiplied to infinity in such a way that there is no fold so small that it is not subdivided by a new fold […] And the tunic cannot be said to be resolved all the way down into points; instead, although some folds are smaller than others to infinity, bodies are always extended and points never become parts, but always remain mere extrema." (*Pacidius*; A 6.3 554-555, LLC 185-187) Leibniz remained faithful to his model of folds throughout his life and infinite flexibility indeed lies at the foundation of his later dynamics (1695, *Specimen of Dynamics*, GM VI 234-254, AG 117-138)

[41] As Arthur pointed out, when Leibniz discovered the Principle of the Conservation of Force in 1678, he now possessed a metaphysical principle that could be used to distinguish between different bodies and protect the consistent existence of the body over time. Leibniz could finally exploit the potential of his model of folds and relate to motion as something imaginary, without thereby denying the existence of matter. In fact, the discovery of force created an inversion in Leibniz' theory of physics, although the solution of the Labyrinth of the Continuum using the model of folds remained intact from November 1676 onward (Arthur 2001a, lxxxv-lxxxvii; Arthur 2014, 88–89). In March 1678, Leibniz solved the paradox of the heap that had bothered him in his *Pacidius* Dialog eighteen months earlier. The paradox makes it difficult to distinguish between poverty and wealth (or between single grains and a heap), since it assumes that the difference is based on a certain point, starting from which poor becomes rich and grains become a heap, or else change does not take place in the world at all. In November 1676, Leibniz arrived at the conclusion that change must be a connection of the last point of the previous situation to the first point of the subsequent situation, namely by means of a leap (29 Oct. – 10 Nov. 1676, *Pacidius to Philalethes*, A 539-540, LLC 155-157). In the absence of a metaphysical principle that preserves the stability of matter, there is no other way to explain change except by means of a leap from one extreme point to another. Leibniz' conclusion in *Pacidius* is that there is no actual continuum and that the material world is always in a chaotic situation of unceasing leaps. However, in March 1678, one month after the discovery of the Law of Conservation of Force, Leibniz claimed that there is a totally different solution to the paradox: a change in quantities or actual magnitudes cannot occur at a particular point in a process since quantities or magnitudes are imaginary and are not in any way differentiated. As a result, the actual continuum appears to be possible and the material world is always only in a situation of continuous changes without an end point (March 1678, *Chrysippus's heap*, A 6.4 69-70, LLC 229-231).

matter are the final destinations of infinite processes in matter and therefore they do not actually exist but rather are an imaginary-fictitious presentation in our mathematical perception. This is the reason that Leibniz declared in the 1680s that "There is no exact and fixed shape in bodies due to the actual subdivision of the continuum to infinity" and that "motion involves something imaginary."[42] The solution of the Labyrinth of Continuum is therefore made possible by means of a disanalogy between the mathematical view of the infinitesimal as fictitious and the physical view of matter as infinitely divisible.[43]

6 Infinity and Folds: A Common Source for both Labyrinths

When Leibniz declared that infinity is the way out of the Labyrinth of Freedom, he in the same breath mentioned the Labyrinth of the Continuum. According to Leibniz, this is the same solution since both labyrinths have the same root, i.e. infinity.

> At last a certain new and unexpected light shined from where I least expected it, namely, from mathematical considerations on the nature of infinity. For there are two labyrinths of the human mind, one concerning the composition of the continuum, and the other concerning the nature of freedom, and they arise from the same source, infinity.[44]

There is also an interesting rhetorical hint to the fact that these two labyrinths have the same solution: both solutions, which are based on his unique approach to infinity, were presented by Leibniz as a response to Descartes' approach. Descartes felt that we are unable to deal with the infinite and precisely because of that we have to accept contradictions and paradoxes related to it. However, Leibniz rejected that view with the claim that not to understand something does not imply that we tolerate contradictions that involve it.[45] Leibniz' response appeared already in November 1676, just

[42] 30 April 1687, *Leibniz to Arnauld*, letter XX, G II 98, AG 86, LA 122. Also: "*There is no determinate shape in actual things*, for none can be appropriate for an infinite number of impressions. And so neither a circle, nor an ellipse, nor any other line we can define exists except in the intellect." (1689, *Primary Truths*, A 6.4 1648, AG 34; 1686, *There is no perfect shape in bodies*, A 6.4. 1613, LLC 297; 1702, *Letter to Queen Sophia Charlotte of Prussia, On What is Independent of Sense and Matter*, G VI 500, AG 187-188)

[43] Arthur 2001a, lxxxv-lxxxviii; Levey 2005, 75–76. Following his criticism of Leibniz' model of folds, Levey believes that the model shows a picture of extreme discontinuity (for details, see Levey's interpretation of Leibniz' model of folds in his 2003 article). But I think, like Arthur's position, that this assessment falls in interpreting Leibniz' mathematics from Cantorian point of view. This is because the infinite division of each piece of matter should not yield an infinite number of singular points. The idea of the folds was translated by Levey into fractals, assuming a Non-divisible singular point to be skipped by an inevitable leap. Leibniz did mention such a leap, but it was only before he identified the force as the critical factor that enables the stability and continuity of matter over time. The infinite flexibility of matter, as can be seen from Leibniz' later physics, does not converge into singular points and cannot be represented by a fractal structure.

[44] 1689?, *On Freedom*, FC 179-180, AG 95; 1704, *Theodicy* (preface), G VI 29, H 53.

[45] "[H]aving contended himself with saying that matter is actually divided into parts smaller than all those we can possible conceive, [Descartes] warns that the things he thinks he has demonstrated ought not to be denied to exist, even if our finite mind cannot grasp how they occur. But it is one thing to explain how something occurs, and another to satisfy the objection and avoid absurdity." (29 Oct. – 10 Nov. 1676, *Pacidius to Philalethes*, A 553-554, LLC 183-185)

before he presented his model of folds as a foundation for the solution of the Labyrinth of the Continuum and again in the late 1680s, in the background to the presentation of the solution of the Labyrinth of Freedom.[46]

As mentioned, Leibniz' model of folds is an important component of the solution of his Labyrinth of the Continuum. In the same way that Leibniz defined a body by means of folds, he also conceptually defined a monad as folded to infinity in a way that is in harmony with the general plan of the world.[47] In view of the fact that "each distinct perception of the soul includes an infinity of confused conceptions which embrace the whole universe", the idea that the "folded" soul "knows the infinite – knows all – but confusedly" makes it possible to combine the model of folds with the solution to the Labyrinth of Freedom. It can be assumed that even if a monad had direct access to the complex algorithm or the spatial plan that orders the entirety of events in the world and thus "to unfold all its folds", it would still not be possible to know whether or not a particular event is indeed included in its complete concept, just as it is impossible to know whether or not a particular infinitesimal movement exists as part of an infinite division of matter.

Why is this so? Because unlike necessary truth, contingent truth must be compossible with the other infinitely many contingent truths that describe the actual world. A complete concept of an individual indirectly includes within it all of the concepts of the other individuals that exist together with it in his world.[48] A complete concept of an individual includes within it, from its unique perspective, the same conceptual space of all things and their relations in the world. Therefore, individuals are considered by Leibniz to be "different universes, which are, nevertheless, only perspectives on a single one, corresponding to the different points of view of each" of them.[49] This conceptual

[46] "That same distinguished philosopher I cited a short while ago [Descartes] preferred to slash through both of these knots with a sword since he either could not solve the problems, or did not want to reveal his view. For in his *Principles of Philosophy* I, art. 40–41, he says that he can easily become entangled in enormous difficulties if we try to reconcile God's preordination with freedom of the will; but, he says, we must refrain from discussing these matters, since we cannot comprehend God's nature. And also, in *Principles of Philosophy* II, art. 35, he says that we should not doubt the infinite divisibility of matter even if we cannot grasp it. But this is not satisfactory, for it is one thing for us not to comprehend something, and quite something else for us to comprehend that it is contradictory." (1689?, *On Freedom*, FC 180-181, AG 95)

[47] "For everything is ordered in things once and for all, with as much order and agreement as possible, since supreme wisdom and goodness can only act with perfect harmony: the present is pregnant with the future; the future can be read in the past; the distant is expressed in the proximate. One could know the beauty of the universe in each soul, if one could unfold all its folds, which only open perceptibly with time. But since each distinct perception of the soul includes an infinity of confused conceptions which embrace the whole universe, the soul itself knows the things it perceives only so far as it has distinct and heightened perceptions; and it has perfection to the extent that it has distinct perceptions. Each soul knows the infinite – knows all – but confusedly." (1714, *Principles of Nature and Grace Based on Reason*, §13, G VI 603, AG 211; 1714, *The Monadology*, §60)

[48] It is important to mention that the relations that exist between all of the individual concepts that make up the concept of the best of all possible worlds (i.e. the concept of the actual world) exist whether or not the individual concepts are complete. These relations also exist between incomplete concepts but in that case, we are prevented from knowing this from the analysis of only a single individual concept. This is because when we speak of the analysis of a contingent truth, it is our intention to reduce it to an identity argument and therefore we are in fact talking about a complete concept "which is so distinct that all its components are distinct" (1709, *New Essays*, book II, ch. xxxi [of complete and incomplete ideas], NE 266; November 1684, *Mediation on Knowledge, Truth and Ideas*, G IV 423, L 292). In contrast, an incomplete concept can be "distinct, and does contain the definition or criteria of the object"; however, "these criteria or components are not all distinctly known as well." (ibid., NE 267). We will discuss the distinct knowledge of the concept in detail in the next section.

[49] 1714, *The Monadology* §57, G VI 617, AG 220.

infinite complexity is consistent with the infinite complexity of the situation of matter. Essentially, Leibniz believed that there is a deep connection between the physics of bodies and the metaphysics of spirits, which forms the basis for the "union between body and soul."[50] Therefore, Leibniz' idea of pre-established harmony has two meanings that correspond to one another: It is based on the existence of a conceptual space that is reflected in each of the monads and it is also applied in actuality by means of a phenomenal space that is created from the simultaneous existence of all bodies. These two spaces correspond to each other and just as there exists a syncategorematic infinity of perspectives on the conceptual space, there also exists a syncategorematic infinity of bodies in the phenomenal space. Together, these two spaces are a valid presentation of the world, where the first presents the world as a complete and ideal spatial plan, while the second presents it as an aggregate of entities. The harmony between these spaces is the result of Leibniz' unity of body and soul and it achieves actual union in each individual, which is based on the 'infinite' force that implements in a continuous, regular and automatic way the infinite details in its internal spatial plan.[51]

It appears that the harmony between the conceptual space of the world and the phenomenal space of the world will make it possible to solve by means of infinity the two labyrinths in the same manner. Essentially, Leibniz formulation of the solution to the Labyrinth of Freedom is similar to that of the model of folds of matter. Moreover, Leibniz even made a clear and deliberate comparison between the infinity associated with the continuum and the infinity related to freedom:

> [T]here is no portion of matter which is not actually subdivided into others; so the parts of any body are actually infinite, and so neither the sun nor any other body can be known perfectly by a creature. Much less can we arrive at the end of our analysis if we seek the mover of each body which is moved, and again the mover of this; for we shall always arrive at smaller bodies without end.[52]

In relation to this text as well as in other places, where Leibniz binds together the infinite division of matter with the infinite analysis of contingent truth, there is an interpretive tendency to understand this as connected with infinite complexity. Indeed,

[50] "And since everything is connected because the plenitude of the world, and since each body acts on every other body, more or less, in proportion to its distance, and is itself affected by the other through reaction, it follows that each monad is a living mirror or a mirror endowed with internal action, which represents the universe from its own point of view and is as ordered as the universe itself. […] Thus there is perfect *harmony* between the perception of the monad and the motions of bodies, pre-established from the first between the system of efficient causes and that of final causes. And in this consists the agreement and the physical union of soul and body, without the one being able to change the laws of the other." (1714, *Principles of Nature and Grace Based on Reason*, §3, G VI 600, AG 207-208)

[51] In view of this, it would have been logical for Leibniz to attribute the infinite force that implements the spatial plan of the world to the world itself, namely to God who operates as the soul of the world. However, Leibniz refrained from doing so. The question of why is the subject of a debate in *The Leibniz Review* between Laurence Carlin (1997), Gregory Brown (1998, 2000), and Richard Arthur (1999, 2001b). I personally believe that the identification of God as the soul of the world is in fact totally consistent with Leibniz' philosophy, though Leibniz refrained from stating it out of a desire to distance himself from Spinozian pantheism. Ambivalence on this issue indeed characterizes the metaphysics of Leibniz from February 1676 until the discovery of the metaphysical force in material bodies in February 1678.

[52] 1686, *Necessary and Contingent Truths*, C 18-19, MP 98; also 1689?, *On Freedom*, FC 181, AG 95.

one cannot deny that an argument focusing on infinite complexity does exist in Leibniz' explanation. But as I have emphasized above, infinite complexity prevents analysis from man but infinite complexity per se is not a compelling reason to block it from God. The fact that things are connected to other things and again to other things in a syncategorematic infinite way is certainly related to the solution but is not the heart of the matter. There is no need to base the solution on the infinite complexity of things, but only on the fact that there is never a final piece of matter, as well as a last step in the analysis of contingent truth. In the quoted text one can see that the analysis of contingent truth never ends as the division of matter never ends. The division of matter of folded matter never ends not because of infinite complexity but because there is never a last fictitious part. Exactly in the same way, I argue that the solution to the labyrinth of freedom lies in the fact that there is no last fictitious step in the analysis of 'folded' contingent truth.

There is complete harmony between the infinite divisibility of matter and the infinite complexity of the contingent subject. If so, is it possible to create an algorithm for infinite complexity of this type? Perhaps not,[53] but Leibniz' expressions give the impression that he believed that a production rule could be formulated for every complexity of every order.[54] It is reasonable to assume that the mathematical achievements that led Leibniz to the development of infinitesimal calculus led him also to believe that all truths, no matter how complex, can be proven, even if the proof is based on an infinite regression.[55]

[53] In his article, Carriero shows that it is not possible to create an algorithm for infinite complexity (Carriero 1993, 24–25). However, I am not certain that such an argument is consistent with Leibniz' philosophy. I think that Leibniz in fact believed that at the foundation of every infinitely complex individual there is an algorithm that organizes the entirety of the attributes and relations of that individual. Since there is compatibility between the definition of the individual and the conceptual space of the world, it is reasonable to assume that the spatial plan of the world can also be formulated using an algorithm: "[E]verything is in conformity with respect to the universal order. This is true to such an extent that not only does nothing completely irregular occurs in the world, but we would not even be able to imagine such a thing. Thus, let us assume, for example, that someone jots down a number of points at random on a piece of paper […]. I maintain that it is possible to find a geometric line whose notion is consistent and uniform, following a certain rule, such that this line passes through all the points in the same order in which the hand jotted them down." (1686, *Discourse on Metaphysics* §6, G IV 432, AG 39; 1686?, *A Specimen of Discoveries*, A 6.4 1619, LLC 310-311) A different question, which is related to the nature of demonstration in Leibniz' philosophy, is whether such an algorithm can be demonstrated and in fact it cannot be.

[54] "It should be no cause for astonishment that I endeavour to elucidate these things by comparisons taken from pure mathematics, where everything proceeds in order, and where it is possible to fathom them by a close contemplation which grants us an enjoyment, so to speak, of the vision of the ideas of God. One may propose a succession or series of numbers perfectly irregular to all appearance, where the numbers increase and diminish variably without the emergence of any order; and yet he who knows the key to the formula, and who understands the origin and the structure of this succession of numbers, will be able to give a rule which, being properly understood, will show that the series is perfectly regular, and that it even has excellent properties. One may make this still more evident in lines. A line may have twists and turns, ups and downs, points of reflexion and points of inflexion, interruptions and other variations, so that one sees neither rhyme nor reason therein, especially when taking into account only a portion of the line; and yet it may be that one can give its equation and construction, wherein a geometrician would find the reason and the fittingness of all these so-called irregularities." (1704, *Theodicy* §242, H 277)

[55] "An analysis of concepts by which we are enabled to arrive at primitive concepts, i.e. at those which are conceived through themselves, does not seem to be in the power of man. But the analysis of truths is more in human power, for we can demonstrate many truths absolutely and reduce them to primitive indemonstrable truths." (1679–1686, *An Introduction to a Secret Encyclopaedia*, C 513-514, MP 8). Note that Leibniz stated this after completing the development of his Infinitesimal calculus.

perhaps this is the reason why Leibniz was surprised to find the solution within infinity. It can be said that the solution was found in spite of the ability to sum infinite series, since Leibniz would have needed time in order to understand what is still preventing the proof of the contingent. Even though the solution to the Labyrinth of Freedom is based on an analogy between mathematics and logic, it must be based simultaneously on a disanalogy between them as well. Just as in the case of the labyrinth of continuum, whose solution is based on a disanalogy between mathematics and physics.

7 The Solution of the Labyrinth of Freedom: Is it Possible to Understand a Fictitious Argument in a Clear and Distinct Way?

It is impossible to argue that Leibniz was not aware of the problem created by the development of his infinitesimal calculus for the solution of the Labyrinth of Freedom. Leibniz was completely aware of the fact that the analogy between the mathematics of incommensurable quantities and the logic of contingent truths might be too strong. If we are capable of describing the nature of an asymptote even though to do so we must deal with an infinite regression, why aren't we able to describe the nature of a contingent truth? In his writings, it appears that Leibniz indeed sought out the difference that nonetheless exists between mathematics of infinite series and logic of infinite analysis, but did not specify what it is.

> So the distinction between necessary and contingent truths is the same as that between lines which meet and asymptotes, or between commensurable and incommensurable numbers. But a difficulty stands before us. we can prove that some line – namely, an asymptote, constantly approaches another, and (also in the case of asymptotes) we can prove that two quantities are equal, by showing what will be the case if the progression is continued as far as one pleases; so human beings also will be able to comprehend contingent truths with certainty. But it must be replied that there is indeed a likeness here, but there is not a complete agreement.[56]

From this point onward, we are left to reconstruct Leibniz' argument since textually it appears that his argument ends here. Still, the connection and resemblance between the two labyrinths invites us to try to solve the Labyrinth of Freedom in the same way Leibniz solved the Labyrinth of the Continuum.

As we saw above, the solution of the Labyrinth of the Continuum is based on the separation between the ideal scope of mathematics and the actual scope of physics. Leibniz recognized that the ideal whole is always lacking parts while the actual parts never produce a whole and therefore he understood that the mathematical solution of the Labyrinth of the Continuum depends on the identification of the infinitesimal as a fiction. On the physical aspect of the labyrinth, the series of infinite changes in matter never ends and therefore matter has no fixed and defined shape but only infinitesimal and blurred

[56] 1686, *General Inquiries about the Analysis of Concepts and Truths* §§135–136; C 388, PLP 77-78.

contours; there is no absolute rest but only infinitesimal movement.[57] Movement, shape or size cannot be exact and differentiated and therefore cannot be perceived clearly and distinctly. Leibniz treated them as fictitious aspects of matter. This is exactly the reason why the internal physical force, which exists at the foundation of a material body and organizes it through an infinite series of situations, never manages to complete its action. The internal force causes motion that continues through infinite intermediate states, all of which are always differentiated, without leaps. The 'last' fictitious state of motion that is supposed to end the movement is only an imaginary mathematical fiction, which helps justify the calculus but cannot be performed by actual force.[58]

Basically, this is also the solution of the Labyrinth of Freedom. Leibniz separated between the analytics of truth and its a priori inclusion of all of its predicates on the one hand and its provability on the other. While the analytics of truth through the concept-containment principle expresses the concept of contingency as ideal and complete, its provability is connected to the infinite analysis that converges step by step to the identity argument which constitutes its boundary. However, in contrast to a mathematical proof that from the start deals with non-differentiated quantities (as a result of the fact that the whole is precedent to the parts), logical demonstration depends on clear and distinct perception of all parts of the deduction. In other words, it is the constraint of logical demonstration that prevents Leibniz from using his mathematical tool.

Let us recall the scale of Leibniz' knowledge and the demands he set for demonstration of truths. In Leibniz' scale of knowledge, the lowest level is the obscure knowledge of visual memory which cannot discern a real distinction between things. Above this level there is the clear recognition that characterizes sensory knowledge and makes it possible to distinguish between flavors, smells and images "one cannot tell them apart in memory but will sometimes tell them apart when they are laid side by side."[59] Better than that is the distinct knowledge that characterizes scientific information and can "discern the true and the false by means of certain tests or marks which make up the definition of gold",[60] for instance. The definition of gold is therefore Obtained for the first time only by distinct knowledge because clear knowledge is still blurred and confused and the distinction it makes is based on examples only. On the

[57] "[R]est. can be considered as an infinitely small velocity or as an infinite slowness. Therefore whatever is true of velocity or slowness is in general should be verifiable also of rest taken in this sense, so that the rule for resting bodies must be considered as a special case of the rule for motion [...]. Likewise equality can be considered as an infinitely small inequality, and inequality can be made to approach quality as closely as we wish." (July 1687, *Letter of Mr. Leibniz on a General Principle Useful in Explaining the Laws of Nature through a Consideration of Divine Wisdom – Reply to Father Malebranche*, G III 51, L 351)

[58] "Since the Cartesians don't understand the use of elastic force in the collision of bodies, they also err in thinking that changes happen through leaps, as if, for example, a body at rest could, in a moment, pass into a state of determinate motion, or as if a body placed in motion could suddenly be reduced to rest, without passing through intermediate degrees of velocity. If elastic force were lacking, then, I confess, what I call the law of continuity, through which leaps are avoided, would not be observed, nor would there be place for other excellent contrivances of the Architect of Nature, contrivances by which the necessity of matter and the beauty of form are united. Moreover, this very elastic force, inherent in every body, shows that there is internal motion in every body as well as a primitive and (so to speak) infinite force, although in collision itself it is limited by derivative force as circumstances demand." (May 1702, *On Body and Force, Against the Cartesians*, G IV 399, AG 255)

[59] 1709, *New Essays*, book II, ch. xxix [of clear and obscure, distinct and confused ideas], NE 255.

[60] 1686, *Discourse on Metaphysics* §24, AG 56.

other hand, scientific recognition that the distinct knowledge yields is based on identifying the clearest signs that gold can be distinguished.

However, the definition provided by distinct knowledge is only a nominal definition in which "one can still doubt whether the notion defined is possible," and therefore "we cannot be certain of the consequences we derive, for if it concealed some contradiction or impossibility, the opposite conclusions could be derived from it."[61] The reason for this is that distinct knowledge reveals all identifying signs that define the concept, but it is not capable of grasping each of them in a distinct way, thus ensuring that the whole complex concept is not based on internal contradiction.

Almost at the top of the scale there is the adequate knowledge, in which "everything that enters into a distinct definition or distinct knowledge is known distinctly, down to the primitive notions."[62] Leibniz referred to the definition of whole numbers as an adequate definition. Adequate knowledge provides knowledge of mathematical or logical precision and is the one that produces demonstrations of necessary truths. In order to go down to the basic and simple roots of the concept, adequate knowledge uses words, symbols, or signs intended to ease the analysis. According to Leibniz "when analysis has been carried to completion, then knowledge is adequate."[63]

Leibniz found the way to justify his calculus using fictitious infinitesimal by which he calculates infinite series and incommensurable sizes. This mathematical breakthrough strengthened Leibniz' ability to determine that all concepts are analytical and include a priori all their predicates. But it is prevented from him when he has to justify or prove this because fictitious information cannot be understood in an adequate way.

In the case of the analysis of truth, which involves a regression to infinity, we do not have available to us the option of viewing the infinitesimal as fictitious. Contingent truths are indeed broken down to infinity due to their complexity, but in contrast to the mathematics of infinite series they are not provable since one cannot ignore the infinitesimal stage in the analysis, by means of which one can arrive at the identity argument. This is exactly the point at which the analogy between mathematics and logic ends. There are mathematical quantities that cannot be perceived in a clear and distinct manner, such as for example roots of negative numbers, but the mathematical process requires their existence. These mathematical entities can be considered to be imaginary in the sense that they belong to the ideal scope of mathematics. Thus, they do not include a contradiction (otherwise they would be excluded from being possibilities) but neither are they candidates for actual realization. In contrast, arguments in the logical process cannot be 'undefined' or 'undifferentiated'. It seems that there is no logical process—at least in the opinion of Leibniz—that has the power to include 'imaginary' arguments that are indeed consistent but also do not describe actual reality.[64]

[61] Ibid., AG 57.
[62] Ibid., AG 56.
[63] Nov. 1684, *Meditation on Knowledge, Truth and Ideas*, G IV 24, AG 25.
[64] Interestingly, Leibniz defined a concept as imaginary when it is possible but does not belong to the conceptual space of the actual world: "[P]ossible ideas become merely chimerical when the idea of actual existence is groundlessly attached to them – as is done by those who think they can find the Philosopher's Stone, and would be done by anyone who thought that there was once a race of centaurs." (1709, *New Essays*, book II, ch. xxx [of real and chimerical ideas], NE 265-266) However, such a situation cannot be part of a process of infinite analysis of contingent truth. The analysis of concepts related to one possible world cannot involve concepts related to a different possible world. Therefore, the analysis of contingent truths about the world cannot include an 'infinitesimal-fictitious' stage that is, by its definition as imaginary, not included in these truths.

Furthermore, even if we assume that a prophet can know with certainty that a future event is already included in the concept of a certain subject (that is, he has access to 'future contingents'), he in any case cannot demonstrate such inclusion by means of analysis. This failure of analysis is not the result of the epistemic deficiency of a created mind, however enlightened. In order to complete an analysis of contingent truth the prophet or the scientist-genius must know the contingent in an adequate way, which means that he knows clearly and distinctly each and every notion that is included in this contingent fact. We can now understand that in order to complete the analysis the prophet must be in an impossible mental state and know in a clear and distinct manner non-differentiated information. Therefore, the lack of an 'imaginary' or 'fictitious' concept in the analysis of contingent truths leads to a situation in which such truths cannot be demonstrated, "otherwise it would be as easy to be a prophet as to be a geometer"[65] and Leibniz assumed it's not.

A geometer or mathematician can diminish as much as he chooses the difference between an infinite series and its limit and ultimately treat it as a negligible error, that is, as if it is zero. But the logician cannot do so because the logical process does not allow him to skip to the identity argument that constitutes the boundary of the analysis and requires him to understand clearly and distinctly the intermediate argument on which he supposedly skips.

Moreover, the reason that the differential is fictitious to Leibniz is that there is no way to know the true components of mathematical size. As mentioned before, the inverse relation between summation and differentiation makes it possible to express each first-order differential series as a sum series of a second-order differential series and vice versa, while blurring the identity of the original series. The differential according to Leibniz is not a quantitative entity but a spatial relation that must be understood as a constant proportion of sizes. However, in the analysis of contingent truths, it is impossible to relate to the stages of the analysis in a purely formal manner. As mentioned earlier, Leibniz did believe that there was an internal spatial relationship between the components of the concepts, a relationship in which Leibniz determined that the truth must be analytic. But this does not mean that the inner components of a statement concerning a contingent truth are to be understood in such a spatial-formal sense that it does not matter which components really makes it. The analysis of contingent truth is still dependent on truth, that is, on the true notions which constitute a complete concept, and cannot be based on fictitious elements, that is, on elements that cannot be known whether they really constitute the truth or not.

This disanalogy between mathematics and logic concerning the use of fictitious information provides Leibniz with the essential gap in order to distinguish between mathematical-geometric necessity and contingent certainty.[66] The solution of the two labyrinths is in fact the same: unlike in mathematics, one cannot perceive the infinitesimal as physical fiction or as logical fiction, since it is impossible to perceive in a

[65] 4/14 July 1686, *Leibniz to Arnauld*, letter X, G II 53, LA 59.

[66] In a paper from the early 1680s, Leibniz allegedly considered fictitiousness as a basis for the solution of the problem of contingency and divine freedom, though not in the form we discussed above (1680–1682?, *On Freedom and Possibility*, Grua 291, AG 22). In that paper, he stated that contingent truths cannot be demonstrated, even though there is no mention of a solution to the Labyrinth of Freedom by means of infinity (the solution took shape for Leibniz during 1686 or somewhat prior to that). To illustrate that contingent truths cannot be demonstrated, Leibniz used the imaginary solution of a quadratic equation. Leibniz pointed to the difference between an impossible solution which must necessarily be ruled out and an imaginary solution which can possibly, and therefore freely, be ruled out. As a result, contingent truths may be unprovable without becoming impossible.

clear and distinct manner real discrete but non-differentiated parts or non-differentiated arguments. Infinite analysis of contingent truth cannot end just as physical changes never end. Just as physical variation has no end even though it is defined by means of a convergent infinite series, so also the infinite analysis of a contingent truth has no end even though such a truth a priori includes all of its predicates.[67] We can say that the solution of the two labyrinths is indeed based on "mathematical considerations on the nature of infinity", but somewhat surprisingly it is the lack of harmony between the labyrinths and the entirety of mathematical considerations that enables the solution of these two labyrinths by means of infinity.

8 Certainty that Is Not Necessitating: God's Intuitive View of Contingency

Finally, we need to ensure that God is unable to prove a contingent truth, according to the solution of the labyrinth of freedom offered here, to verify that the freedom obtained is not an illusion. So how does God perceive a contingent truth? God does not perceive the contingent truth by analysis of its infinite chain of reasons but rather directly and with a single glance.[68] Previously, we showed that the general order of the world is manifested by means of a conceptual space that includes the full relationship between all of the monads that form the world. The general order of the world is also manifested by means of the phenomenal space which includes the aggregate of effects and physical relations between all phenomena in the world. Even though the conceptual space and the phenomenal space are in harmony and ordered as a spatial plan or a general law of the entire world, the monads, who constitute the world and participate in it, refrain from perceiving this general order in a direct manner. They must perceive the general algorithm of the world from their personal point of view, i.e. from within the unique position that they occupy in that spatial plan of the world. This must be a posteriori recognition of the organization of causes in the world, which is characterized as confused and folded within itself. Moreover, this is also a recognition that can never arrive at completeness.[69] God, however, perceives the spatial order of the world in a complete way. This is because he perceives the algorithm of the world's overall order directly[70]:

> God alone has distinct knowledge of the whole, for he is its source. It has been said quite nicely that he is like a center that is everywhere, but that his circumference in nowhere, since all is present to him immediately, without any distance from this center.[71]

[67] Arthur refrained from arguing this claim and emphasizes that there is a complete analogy between infinite analysis of contingent truth and the mathematics of infinite series (Arthur 2014, 96). For this reason, his explanation does not answer the fact that Leibniz' calculus actually allowed him to prove, in a finite number of steps, that there is a sum of infinite series and a way to calculate the size of an incommensurable figure.

[68] "But in contingent propositions one continues the analysis to infinity through reasons for reasons, so that one never has a complete demonstration, though there is always, underneath, a reason for the truth, but the reason is understood completely only by God, who alone traverses the infinite series in one stroke of mind." (1686?, *On Contingency*, Grua 303, AG 28).

[69] 1686, *Necessary and Contingent Truths*, C 17, MP 96.

[70] Ishiguro 1990, 196.

[71] 1714, *Principles of Nature and Grace Based on Reason*, §13, G VI 603, AG 211.

God perceives the contingent truth in a direct way such that all of its infinitely many internal characteristics are perceived by him simultaneously. This claim implies that God perceives the truth by means of the "definite analogy between characters and things" that constitutes "the basis of truth",[72] namely by the truth's internal law of production. One can say that if God indeed perceives any contingent truth directly, then such a truth must have an organizing principle or internal relation between all of its predicates.[73] In other words, without an organizing principle or internal algorithm, even God cannot perceive a contingent truth and thus there is no guarantee that the contingent truth is analytical and a priori includes the endpoint of its analysis.

Seeing the contingent truth of the actual world as an organized information in a spatial plan like an algorithm is helpful and even crucial but the analogy between mathematics and logic must end here. An infinite convergent series does not include the endpoint that limits it, but the series functions as if the boundary is nevertheless included in it, so we used to neglect the fictitious infinitesimal last part. In contrast, Although God is exposed to the internal order of a contingent subject from within, he cannot determine whether the identity argument that represents the limit of the infinite analysis is included within it or not. God also cannot tell, by knowing the algorithm that defines the contingent truth, that an infinite analysis of the subject nonetheless converges to this identity argument as if this identity argument is indeed within it. In short, God is unable to reconstruct the way in which an infinite analysis of the subject arrives at this identity argument, because in order to do so he must perceive an undifferentiated information which is fictitious and imaginary in a clear and distinct manner. God can detour around this by means of an intuitive holistic perception.

Recall that in the analogy that Leibniz constructed between mathematics and logic, he compared the simple, that is rational, knowledge of God to arithmetic and the divine intuition to geometry.[74] The idea behind these comparisons is that God can perceive truth only because he perceives it as a whole. If God had to perceive the truth through its infinitely many components, it would be inaccessible to even him. Geometrically, the area that is obtained from the summation of infinite infinitesimal areas can be identified — even though in principle the mathematical process is infinite — because the whole precedes its undifferentiated parts. A 'geometric' vision of the truth is based on the fact that its infinitely divisible components are undifferentiated and therefore, such vision is not able to know them—'arithmetically'—one after the other. Even God is unable to perceive in a distinct manner undifferentiated, confused and imaginary arguments, and he is unable to 'arithmetically' characterize quantities that can be known only 'geometrically'. If a full analysis of contingent truths is not possible by definition, then God is also unable to follow one argument after another in the course of an infinite analysis until arriving at the last identity argument. This is the reason why

[72] August 1677, *Dialogue*, G VII 193, L 185.

[73] As argued by Nachtomy, the complete concept of a contingent subject, which is composed of an infinite set of its predicates, is defined by a production rule or an algorithm. This allows Leibniz to avoid the paradox related to an infinite number. Due to his intensional perspective on his concept-containment principle, Leibniz can claim that the complete concept of a contingent subject does not include an infinite number of characteristics but rather expresses the regularity or order that generates a syncategorematic infinity of characteristics and ratios (Nachtomy 2007, 63–67).

[74] 1685–1689?, *The Source of Contingent Truths*, C 1-3, AG 98-100.

Leibniz stressed that God does not perceive the truth by means of investigating all of its predicates but rather by means of intuition.[75]

How can such an intuitive perception be characterized? One can perhaps say that God intuitively perceives the algorithm that underlies all of the contingent truths just as we intuitively perceive the primary truths. When we look at an identity argument we recognize immediately that it is a priori valid even though we are unable to prove it. Also primary a posteriori truths, such as Descartes' Cogito Argument, which underlie the contingent truths, are recognized by us directly. These truths are "inner experiences that are immediate with the immediateness of feeling"[76] and we also know them with one glance, in one stroke. We have no need of a proof in order to know that they are valid. This is the way that God perceives not only simple concepts and obvious equivalence relations but also infinitely complex concepts. He perceives all at once, with one glance, the infinity of situations that are ordered by the spatial program of the world. God knows directly, a priori, that this algorithm does not contain an internal contradiction,[77] but even he cannot guarantee it by means of an analysis.[78]

[75] "For necessary truths can be resolved into such as are identical, as commensurable quantities can be brought to a common measure; but in contingent truth, as in surd numbers, the resolution proceed to infinity without ever terminating. And so the certainty of and perfect reason for contingent truths is known only to God, who grasps the infinite in one intuition." (Circa 1686–1688, *A Specimen of Discoveries*, A 6.4 1615, G VII 309, trans. by Russell 1900, 221–222, LLC 305).

[76] 1709, *New Essays*, book IV, ch. ii [The degrees of our knowledge], NE 367.

[77] It seems to me that only God can know a priori that a contingent truth is possible, i.e. that it does not include internal contradiction, since only he sees it as a whole, while no one, including God, can analyze all its components one after another. Nevertheless, inspired by an overly strong analogy between mathematics and logic, Arthur declares that Leibniz has proved that contingent truth is possible: "by defining a contingent truth as one in which, although the predicate is in the concept of the subject, it is impossible for this to be demonstrated, Leibniz has proved its possibility, a possibility revealed to him by the analogy with incommensurables." (Arthur 2014, 96) In fact, precisely because of this analogy, Leibniz concluded that the mathematical demonstration of the calculation of incommensurable sizes must be different from the demonstration of contingent truths. Otherwise contingency will collapse into mathematical necessity.

[78] A contingent truth is not necessary and therefore its negation does not imply a contradiction (1686, *Discourse on Metaphysics* §13, AG 45). In other words, this is a truth that can never be demonstrated to include the argument that contradicts it. But to the same extent it cannot be demonstrated that it does not include such an argument. Therefore, an analysis of contingent truth cannot guarantee that it is possible at all. Such an analysis is based on a nominal definition of the object of the investigation and we need an *a posterior* confirmation that a particular fact is indeed possible. Therefore, certainty with respect to the validity of a contingent truth is not due to the knowledge that this truth does not include an argument that contradicts it but rather is due to the lack of knowledge as to whether or not this truth includes an argument that contradicts it. This is because Leibniz was ready to accept that possibility is not only everything that necessarily does not imply a contradiction, but also everything that does not necessarily imply a contradiction. Such an argument also lies at the foundation of Leibniz' ontological proof which functioned as his formal proof that the concept of God is possible (18–21 Nov.? 1676, *That a Most Perfect Being Exists*, A 6.3 578-579, PDSR 101-103; 1704, *Theodicy* §23, H 88). As we know, although Leibniz was concerned that a thought which seems possible is not really so (for example, the notion of an 'infinite number'), he was also aware of the fact that proof of possibility was almost always limited to a nominal or "merely real and nothing more" definitions (1686, *Discourse on metaphysics* §24, AG 57). In light of this, it is somewhat surprising to find that certainty is related to the fact that we cannot absolutely know whether or not a concept is possible. This certainty should be referred to as an intuitive certainty that cannot be proven, since intuition perceives the entirety (the whole which has no parts) while a proof perceives the complex and consistent process of the parts. We intuitively perceive simple concepts only and we are unable to prove or justify such concepts in the same way that we do in the case of complex concepts that do not include an internal contradiction. On the other hand, God can also perceive with an intuitive glance infinitely complex concepts; in other words, he can see them as if they were simple although even he cannot prove that there is no internal contradiction in such concepts. In a similar way, I have claimed that the process of intuitive perception clearly cannot be complex but can only be simple; it therefore fails in providing a priori proof of the existence of God (Lison 2017).

To sum up: God intuitively perceives the contingent by means of a direct, holistic and unmediated approach to the fundamental algorithm of the world. God of course knows with certainty that this algorithm is possible but even he is unable to prove this one stage after another until arriving at an identity argument, which serves as the limit of that analysis. Even he cannot perceive a fictitious logical argument as a distinct one.

9 Conclusion

In order to distinguish between necessary and contingent truths, Leibniz certainly made use of his new mathematical developments and he does not rely only on the way in which the Greeks dealt with irrational numbers. Only thanks to his calculus could Leibniz have justified his argument that all truths—including contingent ones—are analytical. After all, based on the Greeks' knowledge, one cannot know whether irrational numbers are subject to rational thinking. Infinitesimal calculus provided Leibniz with the idea that an infinitely complex set of relations can be calculated by means of a production rule or algorithm and in this way the idea of the inclusion of the predicate within the subject can be perceived as possible. Nonetheless, the proof that an idea is possible is mathematical, namely it is based on the existence of fictitious infinitesimal quantities that can be ignored in order to complete the infinite regression. Such a proof is not available in the infinite analysis of contingent truths. Leibniz grappled with the problem for a long time until he noticed that the solution lies in the infinite. Reconstruction of his solution shows that the weak point of the mathematical proof exploited in order to solve the Labyrinth of Freedom. Even God can perceive only intuitively the algorithm that is at the foundation of a contingent truth without being able to provide a proof, and that is why contingent certainty is not necessary. Indeed, this spatial plan of the contingent world is accessible to God only while we, due to our conscious limitations, cannot all at once "unfold all" of our "folds which only open perceptibly with time" and thus we can "know the infinite, know all, but confusedly." But this epistemic limitation cannot obscure the fundamental limitation due to which the distinction between necessity and contingency produces real freedom and not just the appearance of freedom. After all, to conceive in a clear and distinct manner fictitious and non-differentiated logical arguments – this even God cannot do!

Abbreviations

A	G. W. Leibniz. *Sämtliche Schriften und Briefe*. Ed. by the Deutsche Akademie der Wissenschaften.Multiple vols. in 7 series. Darmstadt/Leipzig/ Berlin: Akademie Verlag, 1923-.
AG	G. W. Leibniz. *Philosophical Essays*. Ed. and trans. by R. Ariew, and D. Garber. Indianapolis and Cambridge: Hackett, 1989.
C	*Opuscules et Fragments inedits de Leibniz*. Ed. by L. Couturat. Paris, 1903. Reprinted Hildesheim, 1961.
Child	*The Early Mathematical Manuscripts of Leibniz*. Ed. and trans. by J. M. Child. Chicago: Open Court 1920. Reprinted New York: Dover, 2005.

FC	*Nouvelles Lettres et Opuscules de Leibniz*. Ed. by A. Foucher de Careil. Paris: Auguste Durand, 1857. Reprinted Hildesheim: Olms, 1962.
G	*Die Philosophischen Schriften von G. W. Leibniz*. Ed. by C. I. Gerhardt. 7 vols. Hildesheim: Olms, 1875. Reprinted 1978.
GM	G. W. Leibniz. *Mathematische Schriften*. Ed. by C. I. Gerhardt. 7 vols. Hildesheim: Olms, 1854. Reprinted 1975.
Grua	G. W. Leibniz. *Texts inedits*. Ed. by Gaston Grua. 2 vols. Paris, 1948
H	G. W. Leibniz. *Theodicy: Essays on the Goodness of God the Freedom of Man and the Origin of Evil*. Ed. by F. Austin, trans. by E. M. Huggard. New Haven, Yale University Press, 1952.
L	G. W. Leibniz. *Philosophical Papers and Letters* (2nd edition). Ed. L. E. Loemker. Dordrecht: D. Reidel, 1969.
LA	*The Leibniz-Arnauld Correspondence*. Ed. and trans. by H. T. Mason. Manchester: Manchester University Press, 1967.
LH	*Die Leibniz-Handschriften der koniglichen offentlichen bibliothek zu Hannover*. Ed. by E. Bodemann. Hannover, 1895. Reprinted Hildesheim, 1966.
LLC	G. W. Leibniz. *The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672-1686*. Ed., trans. and with an introduction by R. T. W. Arthur. New Haven: Yale University Press, 2001.
LR	*The Leibniz-Des Bosses Correspondence*. Ed., trans. and with an introduction by B. C. Look, and D. Rutherford. New Haven and London: Yale University Press, 2007.
MP	G. W. Leibniz. *Philosophical Writings*. Ed. and trans. by M. Morris and G. H. R. Parkinson. London: Dent, 1973.
NE	G. W. G. W. Leibniz. *New Essays on Human Understanding*. Ed. and trans. by P. Remnant, and J. Bennett. Cambridge: Cambridge University Press, 1981.
PDSR	G. W. Leibniz. *De Summa Rerum: Metaphysical Papers 1675-1676*. Ed., trans. and with an introduction by G. H. R. Parkinson. New Haven and London: Yale University Press, 1992.
PLP	G. W. Leibniz. *Logical Papers*. Ed. and trans. by G. H. R. Parkinson. Oxford: Clarendon Press, 1966.

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