The Philosophical Assumptions Underlying Leibniz's Use of the Diagonal Paradox in 1672

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Zusammenfassung

Im November 1672 schloss Leibniz, dass ein Kontinuum nicht aus Punkten besteht. Der Beweis, der als Diagonal-Paradox Bekanntheit erlangte, wurde von Leibniz vorgebracht, nachdem er die Existenz einer unendlichen Zahl verneint hatte. Vor kurzem haben mehrere Kommentatoren darzustellen versucht, dass der Leibniz'sche Beweis, unter dem Aspekt von Cantors Mengenlehre und seiner Lehre von den Kardinalzahlen gesehen, nicht stichhaltig sei. In diesem Artikel unternehme ich den Versuch, die philosophischen Annahmen, denen Leibniz' Gebrauch des Diagonal-Paradox unterliegt, offenzulegen, um zu zeigen, dass eine solche Kritik unmöglich ist. Die Kritik gründet sich auf die Forderung, zwischen zwei Wegen, Größen miteinander zu vergleichen, zu unterscheiden; jedoch hatte Leibniz solch eine Unterscheidung schon im Sinn, die er 1672 aber vermeiden wollte. Gegen Ende 1670 dachte Leibniz, dass ein Weg existiere, ein Kontinuum aus Punkten durch eine Unterscheidung zwischen der Ausdehnung eines Körpers und seiner Größe zusammenzusetzen. Diese Unterscheidung erlaubte es Leibniz ebenfalls, verschiedene Größen gleichzusetzen und somit dem Diagonal-Paradox auszuweichen. Im November 1672 versuchte Leibniz jedoch, diese Unterscheidung zwischen Ausdehnung und Größe zu vermeiden, weil er davon überzeugt war, dass eine unendliche Zahl nicht möglich ist, was ihn dazu brachte, den Punkt als einen Bestandteil des Kontinuums zu verneinen.

In this article, it is my intention to discuss the philosophical assumptions underlying the proof Leibniz provided in November 1672 that a continuum is not made up of points. This proof, which became known as the Diagonal Paradox, was offered by Leibniz after denying the existence of an infinite number. Several commentators had claimed, based on Georg Cantor's set theory from the late 19th century, that neither the notion of an infinite whole nor that of an infinitesimally minimal point as a component of a continuum can be negated. I will try to show that this critique is incorrect from a historical perspective since it does not take into account the philosophical starting point from which Leibniz made his claims.

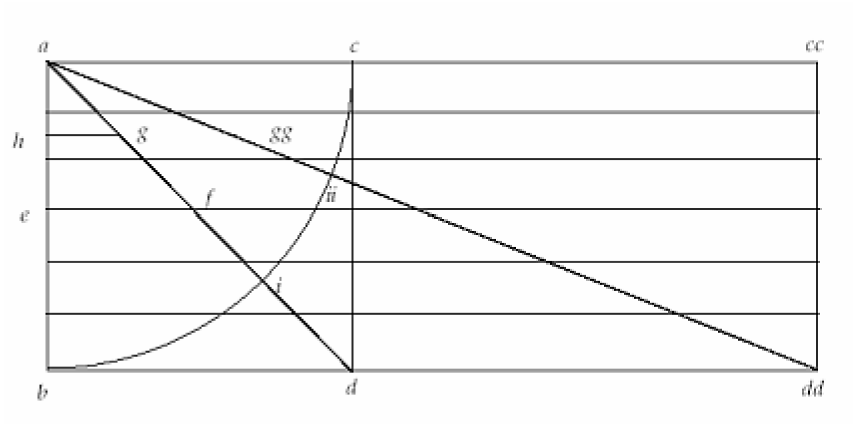
When Leibniz arrived in Paris in March 1672, he began taking an interest in infinite series and, as the result of a recommendation from Christian Huygens, became exposed to the writings of Galilei Galileo. Leibniz was impressed by Galileo's claims but came to different conclusions from those reached by him. Galileo claimed that if one compares the infinite series of natural and squared numbers one arrives at a paradox. On the one hand, every natural number can be calculated as the root of a squared number and therefore for every natural number, one can find a squared number and conclude that the infinite quantities of natural and squared numbers are equal. On the other hand, it is clear that the series of squared numbers constitutes only a part of the series of natural numbers. Indeed, the proportion of squared numbers declines as the series of natural numbers increases in length (squared numbers account for 10% of the first 100 natural numbers, 1% of the first 10,000 and 0.1% of the first million). Therefore, the infinite quantity of natural numbers cannot be equal to the infinite quantity of squared numbers. Galileo's conclusion from this paradox is that infinite magnitudes cannot be compared. In other words, the operators 'equal to', 'smaller than' and 'greater than' have no validity with respect to infinite magnitudes[1]. Leibniz, on the other hand, claimed that Euclid's axiom – in which the whole is larger than its part – has been proven and therefore opposed any challenge to these operators, which serve as the basis for the definition of the part-whole axiom. Instead, Leibniz claimed that the comparison of infinite magnitudes is problematic because of the magnitudes themselves. An infinite magnitude is itself a contradiction between a whole and its part. Therefore, Leibniz denied the existence of an infinite number.

"There is no maximum in things; or what is the same thing, the infinite number of all unities is not one whole, but is comparable to nothing [...] there are as many numbers as there are square numbers, that is, the number of numbers is equal to the number of squares, the whole to the part, which is absurd"[2].

In the same paper, Leibniz also formulates a proof that rejects the construction of a continuum from points. This is a variation of the claim made from series of numbers, which Leibniz uses to prove that an infinite number is not possible. In the proof, which came to be known as Galileo's Diagonal or the Diagonal Paradox, Leibniz shows that if a continuum were indeed composed of an infinite number of points, then two lines of different length would contain the same infinite number of points. In other words, two lines that are different in length should have the same number of components. Leibniz claims that this is impossible and therefore rejects the idea that a continuum is made up of points:

1 "This is one of the difficulties which arise when we attempt, with our finite mind, to discuss the infinite, assigning to it those property which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinities as being the one greater or less than or equal to another […]" (*G. Galileo: Dialogues Concerning Two New Sciences*; trans. by H. Crew and A. de Salvio, with an introduction by A. Favaro, New York 1914, p. 31); "The attributes 'equal', 'greater', and 'less' are not applicable to infinite, but only to finite quantities" (*ibid*., p. 32).

2 Nov. 1672 – Jan. 1673; "De minimo et maximo. De corporibus et mentibus" ("On Maximum and Minimum. On Bodies and Minds"); A VI, 3, 98; quoted from *G. W. Leibniz: The Labyrinth of the Continuum: Writings on the Continuum Problem 1672-1686* (= *The Yale Leibniz*), translated, edited and with introduction by R. T. W. Arthur, New Haven 2001, p. 13 (hereafter: LLC).



"*There is no minimum, or indivisible, in space and body*. For if there is an indivisible in space or body, there will be one in the line *ab*. If there is one in the line *ab*, there will be indivisibles in it everywhere. Moreover, every indivisible point can be understood as the indivisible boundary of a line. So let us understand infinitely many lines parallel to each other, and perpendicularly to *ab*, to be drawn from *ab* to *cd*. Now no point can be assigned in the transverse line or diagonal *ad* which does not fall on one of the infinitely many parallel lines extending perpendicularly from *ab* […] therefore the line *ad* will have as many indivisible points as there are parallel lines extending from *ab*, i. e. as many as there are indivisible points in the line *ab*. Therefore there are as many indivisible points in *ad* as in *ab*. Let us assume in *ad* a line *ai* equal to *ab* […] there will be as many indivisible points in *ai* as in *ad*. Therefore there will be no points in the difference between *ai* and *ad*, namely, in *id*, which is absurd"[3].

Such a one-to-one correspondence between the points of the diagonal and the points of one side of the square implies that each line has the same infinite number of components. However, the diagonal and the side are nonetheless different in magnitude. The diagonal is the larger of the two and therefore a segment of it is equal to the side, which means that only a segment of the diagonal has the same infinite number of points as the side of the square. How can it then be claimed that the diagonal and a segment of it have the same infinite number of points? How can different lines be equal?

There are several possibilities for resolving this paradox but only one that Leibniz is willing to accept. Leibniz is not interested in denying the Euclidian principle that the whole is larger than its part. Neither is he willing to claim that a continuum is not in any way related to infinity. What remains is to deny that a line is made up of points. At this point, Leibniz arrives at the starting point in the development of his infinitesimal calculus in which a continuum is made of segments that can be infinitely divided, with neither minimal beginning nor maximal end.

3 *Ibid*.; A VI, 3, 97-98; LLC, pp. 9-11. It is worth noting that Leibniz was certainly not the first to use the 'Diagonal Paradox'. William of Ockham, for instance, also used this argument to deny the existence of points in the continuum (*William of Ockham: Quodlibetal Questions*, part I, question 9, translated by A. J. Freddoso and F. E. Kelley, 2 vols., New Haven – London 1991, vol. I, pp. 46-48).

However, commentators point to another way of resolving the paradox that Leibniz did not take into account and which both make his conclusions unnecessary and preserve the line as a collection of points.

In his book on Leibniz, Bertrand Russell viewed the proof ruling out an infinite whole in a positive light and even claimed that it is the best way to avoid the contradictions created by infinity[4]. Nonetheless, in light of Cantor's set theory of 1882, Russell later claimed that Leibniz's argument is problematic since he was unaware that there are two meanings for the mathematical operators 'smaller than', 'greater than' and 'equal to'[5]. According to the first, two magnitudes can be compared using their total quantities (or using the extreme members of the series which define the sum of the finite series). According to the second, sets can be compared through a correspondence between their members. A one-to-one correspondence between the series of natural and square numbers demonstrates that their infinite magnitudes are equal even though one is the subset of the other. In fact, for Cantor, disproving Euclid's part-whole axiom provides the fundamental definition of an infinite magnitude since in the case of infinite magnitudes the whole *is* equal to its part. Thus, Leibniz's conclusion regarding the impossibility of an infinite number can be rejected[6].

Other commentators make use of this insight in order to disprove not only Leibniz's rejection of the possibility of an infinite number but also the claim made by Leibniz that a continuum is not made up of points. Samuel Levey and Gregory Brown noticed that Leibniz's proof is based on a congruence criterion of 'greater than' and not on a one-to-one correspondence criterion of 'greater than'. Thus, according to them, there exists an additional implicit assumption in Leibniz's proof that can be rejected: equality in the quantity of points making up

4 "This principle [that infinite aggregates have no number] is perhaps one of the best ways of escaping from the antinomy of infinite number" (B. Russell: *A Critical Exposition of the Philosophy of Leibniz*, London 1900 (reprinted 1937), p. 117).

5 "This property [that the number of natural numbers is the same as the number of even natural numbers] was used by Leibniz (and many others) as a proof that the infinite numbers are impossible; it was thought self-contradictory that 'the part should be equal to the whole.' But this is one of those phrases that depend for their plausibility upon an unperceived vagueness: the word 'equal' has many meaning, but if it is taken to mean what we have called 'similar' [i. e., standing in one-one correspondence], there is no contradiction, since an infinite collection can perfectly well have parts similar to itself" (B. Russell: *Introduction to Mathematical Philosophy*, London 1918, p. 80).

6 The explanation of Russell's conflicting views of Leibniz's rejection of the existence of infinite numbers is quite simple. Russell claimed, as did Cantor, that infinite quantities are possible with various scales or powers and therefore that Leibniz's rejection of the notion of an infinite number is incorrect. In contrast, with regard to absolute infinity, Russell adopted the basic idea of Leibniz's rejection of the possibility of an infinite number as the 'number of all numbers'. Following Cantor, Russell also recognized that the 'number of all numbers' or the 'set of all sets' contains a contradiction since it is inconsistent and therefore not possible.

two lines implies equality in the lines' total magnitudes[7]. Leibniz's conclusion, which denies that a continuous line is made up of points due to the part-whole axiom, is incorrect since a correspondence of members shows that the whole is *not* larger than its part[8]. The distinction between various meanings of the 'greater than' operator makes it possible for a line to be made up of points and at the same time preserves the part-whole axiom for finite magnitudes.

Can Leibniz's position be defended against this critique? On the one hand, there is something anachronistic in claims based on Cantor's theories of the late 19th century against the deliberations of a 17th century philosopher. On the other hand, what is being considered is not a contingent context-dependent claim but rather a necessary a-historical claim of mathematical truth. According to Richard Arthur, the critique is unable to completely negate Leibniz's claim since Leibniz's and Cantor's positions are logically equivalent; thus one cannot be preferred over the other. Leibniz's denial of the possibility of an infinite number is based on accepting the part-whole principle for infinite quantities while Cantor's negation of this principle for infinite quantities is related to his recognition of the existence of infinite numbers[9]. However, this conclusion is problematic: to the same extent that a critique based on Cantor cannot disprove Leibniz, Leibniz's proofs cannot disprove later claims based on Cantor. Thus, Leibniz cannot claim that an infinite number is not possible and necessarily implies a contradiction since the variation proposed by Cantor does in fact make it possible[10]. Even so, the critique forwarded by Russell, Levey, Brown and others makes a relevant point only with respect to the way in which Leibniz justifies his claim. Apparently, it is the absolute validity that Leibniz ascribes to his claim that is in error since Leibniz's argument is in the end contingent.

7 Later on Leibniz himself explicitly say that "two things all of whose parts are equal, are equal" (Dec. 1675, "De materia, de motu, de minimis, de continuo" ("Matter, Motion, Minima and the Continuum"); A VI, 3, 469; LLC, p. 39)

8 "It is entirely possible that the part-whole axiom simply invokes a criterion for 'less than' distinct from the one that the part and whole of the line fail to satisfy [...] perhaps it never occurs to Leibniz to weigh carefully the idea that 'less than' might be polysemous, invoking different criteria in different contexts" (S. Levey: "Leibniz on Mathematics and the Actually Infinite Division of Matter", in: *The Philosophical Review* 107 (1998), pp. 49-96, p. 62; see also G. Brown: "Leibniz on Wholes, Unities, and Infinite Number", in: *The Leibniz Review* 10 (2000), pp. 21-51, p. 22).

9 "Here [Leibniz] identifies two candidates for rejection: (W) that in the infinite the whole is greater than the part, and (C) that an infinite collection (such as the set of all numbers) is a whole or unity [...] Leibniz upholds W, and this leads him to reject C. Cantor upholds C, and this leads him to reject W [...] The success of Cantor's theory counts against Leibniz's choice [...] but it does nothing to show whether Leibniz's theory is inconsistent [...] W→ ~C (Leibniz) is equivalent to C→ ~W (Cantor)" (R. T. W. Arthur: "Leibniz on Infinite Number, Infinite Whole, and the Whole World: A Reply to Gregory Brown", in: *The Leibniz Review* 11 (2001), pp. 103-116, pp. 103-104).

10 G. Brown: "Leibniz's Mathematical Argument Against a Soul of the World", in: *British Journal for History of Philosophy* 13 (2005), pp. 449-488, p. 486.

However, this critique cannot disprove the justification of the content of the claim itself.

The question of whether or not a mathematical truth can be considered as a contingent claim is crucial and I shall return to it at the end of the paper. At this point, it is important to distinguish between the use Leibniz makes of the Diagonal Paradox and his final conclusion concerning the infinite. It seems to me that on historical grounds one can provide a more comprehensive defense for the early Leibniz, if not the late one. It is possible that the critique of Leibniz's attempt to claim that a continuum is not composed of points ignores his philosophical assumptions prior to the Diagonal Paradox. As mentioned, Leibniz's claim is based on the implicit assumption that an equal correspondence between components implies equality in the magnitudes of what they make up. Levey and Brown expected that Leibniz would distinguish between the two methods for comparing magnitudes and therefore would arrive at the conclusion that equal quantities of components do not necessarily imply equal total magnitudes of sets (i. e. the conclusion that lines are made up of points does not prevent lines being of different magnitudes). However, Leibniz did in fact reach such a conclusion. In fact, it is exactly this kind of thinking that he wished to avoid.

Two years earlier, in the winter of 1670, Leibniz wrote the *Theoria Motus Abstraci* (hereafter: *TMA*). This is a complicated and far-reaching theory in which Leibniz explicitly agrees with the idea that a continuum is made up of points[11]. Although Leibniz is already aware at this stage of the paradoxes that arise in composing a continuum from an infinite number of dimensionless points, at this point in time Leibniz was still not convinced that a maximal infinite number is not possible and therefore neither was he interested in totally ruling out the existence of a minimal point as the elementary component of a continuum. Therefore, Leibniz changed the accepted definition of the point.

In contrast to Euclid's definition of a point as having no magnitude or parts, Leibniz claims that a point must have parts. First of all, a dimensionless point cannot be located in space. Such a point cannot be referred to in a spatial context and thus its definition as a point in space is contradictory. Second, the dimensionless minimal point creates the paradox of the whole being equal to its part:

"*There is no minimum in space or body*, that is, there is nothing which has no magnitude or part. For such a thing has no situation, since whatever is situated somewhere can be touched by

11 "The theory of abstract motion explains the hitherto unresolved difficulties of continuous composition; confirms the geometry of indivisibles and arithmetic of infinities; it shows that there is nothing in the realm of nature without parts; that the parts of any continuum are in fact infinite; that the theory of angles is that of the quantities of unextended bodies; that motion is stronger than motion, and endeavor is stronger than endeavor – however, endeavor is instantaneous motion through a point, and so a point may be greater than a point" (Leibniz to Oldenburg, 29 April 1671; quoted from *The Correspondence of Henry Oldenburg*, edited and translated by A. R. Hall and M. B. Hall, 13 vols., Madison – London 1965-1986, vol. 8: *1671-1672*, Madison 1971, p. 26).

several things simultaneously that are not touching each other, and would thus have several faces; nor can a minimum be supposed without it following that the whole has as many minima as the part, which implies a contradiction"[12].

Inspired by Hobbes[13], Leibniz relates to his new point as a kind of conatus, i. e. a momentary and localized tendency to move that makes up a continuous movement, which is itself made up of parts. This means that one point can be larger or smaller than another.

However, Leibniz then immediately proves that the elementary point of a continuum must be indivisible and unextended. In light of Zeno's paradoxes, Leibniz claims that infinite division does not make it possible for a beginning to exist. If the beginning of a continuum is taken as given, it must be indivisible.

"*There are indivisibles or unextended things*, otherwise neither the beginning nor the end of a motion or body is intelligible [...] Nothing is a beginning from which something on the right can be taken away. But that from which nothing having extension can be taken away is unextended. Therefore the beginning of a body, space, motion, or time (namely, a point, an endeavour, or an instant) is either nothing, which is absurd, or is unextended, which was to be demonstrated"[14].

Leibniz's new point is defined as a hybrid between Euclid's indivisible and dimensionless point and Thomas Hobbes' point which possesses internal parts. Thus, Leibniz's point is indivisible and unextended but also has parts and magnitude:

"*A point is not that which has no part*, nor that whose part is not considered; but that which *has no extension*, i. e. whose parts are indistant, whose magnitude is inconsiderable, unassignable, is smaller than can be expressed by a ratio to another sensible magnitude unless the ratio is infinite, smaller than any ratio that can be given"[15].

How is this unique point, which White called a non-standard or non-Archimedean line[16], meant to be understood? Leibniz seeks to understand a continuum through the concept of infinity. Many 17th century mathematicians considered the continuum as an ongoing movement with an infinite number of parts. Galilei Galileo, Bonaventure Cavalieri, Gregory of Saint Vincent and John Wallis posited that the continuum is made up of infinite, non-quanta, dimensionless and indivisible points with a magnitude of zero[17]. Accordingly, they all had difficulty justifying their calculation. Leibniz, inspired by Hobbes,

12 Winter 1670-1671, *Theoria Motus Abstracti* (hereafter: *TMA*), "Fundamenta praedemonstrabilia" ("Predemonstrable Foundations") § 3; A VI, 2, 264; LLC, p. 339.

13 *Leibniz to Hobbes*, 22 July 1670; GP VII, 573; cf. D. M. Jesseph: "Leibniz on the Foundation of the Calculus: The Question of the Reality of Infinitesimal Magnitudes", in: *Perspective on Science* 6 (1998), pp. 6-40, p. 14.

14 *TMA* § 4; A VI, 2, 264; LLC, p. 339.

15 *TMA* § 5; A VI, 2, 264-265; LLC, pp. 339-340.

16 M. J. White: "The Foundations of the Calculus and the Conceptual Analysis of Motion: The Case of the Early Leibniz (1670-1676)", in: *Pacific Philosophical Quarterly* 73 (1992), pp. 283-313, pp. 295-296.

17 E. Knobloch: "Galileo and Leibniz: Different Approaches to Infinity", in: *Archive for History of Exact Sciences* 54 (1999), pp. 87-99, p. 90, and Jesseph: "Leibniz on the Foundation of the Calculus" (see note 13), pp. 23-26.

claimed that the beginning of movement is not rest but a momentary and pinpoint tendency to move. The relation between rest and movement is identical to the ratio between 0 and 1 and according to Leibniz in the *TMA* this is too extreme a difference. In his opinion, unextended minimal points cannot make up a continuous line just as an infinite number of zeros cannot make up a whole number. On the other hand, the ratio between the Hobbesian point and the continuum is not infinite at all since it is a variation of an atom, finite in its size. In the *TMA*, Leibniz is looking for an infinite ratio with which to understand the continuum; thus he perceives the continuum as the ratio between 1 and ∞ rather than the extreme ratio between 0 and 1:

"The ratio of rest to motion is not that of a point to space, but that of nothing to one [...]"[18];

"Endeavor is to motion as a point is to space, i. e. as one to infinity, for it is the beginning and end of motion"[19].

The elementary component of a continuum is not 0 but 1; in other words, it is an indivisible and unextended unit, which is not a dimensionless minima because it has a magnitude determined by its parts.

According to Descartes' intuition, movement can be decomposed into vectors that represent only one direction, such that the elementary movement is a straight one[20]. In contrast to Descartes, Leibniz in the *TMA* perceives the momentary moment or the conatus as made up of parts or 'signs' in space in a way that allows for one moment to be larger than another[21]. In one conatus there may be tendencies in a number of directions and such a conatus is larger than another with a tendency in only one direction. Therefore, straight movement in the *TMA* is no more elementary than circular movement and "different conatus mixed one with another by the least particles produce motions of a new kind"[22]. In any case, Leibniz is persistent in viewing all these 'different' conatus, defined as *partes extra partes*, as indivisible and unextended points in order to claim that their infinite quantity makes up a continuum:

18 *TMA* § 6; A VI, 2, 265; LLC, p. 340.

19 *TMA* § 10; A VI, 2, 265; LLC, p. 340.

20 "Of all motions, only a motion in a straight line is entirely simple and has a nature which may be wholly grasped in an instant. For in order to conceive such motion it suffices to think that a body is in the process of moving in a certain direction, and that this is the case at each determinable instant during the time it is moving. By contrast, in order to conceive circular motion, or any other possible motion, it is necessary to consider at least two of its instants, or rather two of its parts, and the relation between them […] I am not saying that rectilinear motion can take place in an instant, but only that everything required to produce it is present in bodies at each instant which might be determined while they are moving, whereas not everything required to produce circular motion is present" (*The Philosophical Writings of Descartes*, edited by J. Cottingham, R. Stoothoff, and D. Murdoch, 2 vols., Cambridge 1985, vol. I, chap. 7: "The World", pp. 96-97).

21 *TMA* § 18; A VI, 2, 266; LLC, p. 342.

22 Leibniz to Oldenburg, 29 April 1671; quoted from *The Correspondence of Henry Oldenburg* (see note 11), vol. 8, p. 26.

"One point of a moving body in the time of its endeavor, i. e. in a time smaller than can be given, *is in several* places or *points of space*, that is, it will fill a part of space greater than itself, or greater than it fills when it is at rest, or moving more slowly, or endeavoring in only one direction; yet this part of space is still unassignable, or consists in a point, although the ratio of a point of a body (or of the point it fills when at rest) to the point of space it fills when moving, is as an angle of contact to a rectilinear angle, or as a point to a line"[23].

At this point, it is worth considering the difference between extension and magnitude for Leibniz in the *TMA*. The definition of a point as having parts but not being divisible and as having magnitude but not extension implies a distinction between the quantity of internal parts and the total space these parts occupy. The quantity of parts in an indivisible point cannot increase or decrease the total spatial size of a point, i. e. its extension. Why is this so? Because magnitude and extension are parallel concepts from Leibniz's perspective in the *TMA*. Leibniz relates to extension, due to the Cartesian approach, as a kind of taking-up of space that takes place in a continuous manner and without parts. In contrast, magnitude is defined as the sum or quantity of internal parts[24]. At a later stage, Leibniz comes to the conclusion that magnitude can be measured only by parts that are abstract and fixed in magnitude[25]. As such, these ideal parts do not actually make up the object's magnitude. However, Leibniz tries in the *TMA* to redefine the point as both indivisible and possessing parts and therefore the parts of a point determine its actual magnitude and are not abstract. Thus, in defining a body we have the simultaneous use of both its continuous extension and its magnitude based on infinite actual parts. This is particularly interesting in the case of a point, which can be characterized by the absence of continuous extension and at the same time by the existence of parts and magnitude.

In the absence of parts, the size of extension of a component in a continuum is only determined through an infinite fixed ratio between it and the continuum as a whole. Since for every size of extension, a point will always be considered

23 *TMA* § 13; A VI, 2, 265; LLC, p. 340.

24 "*Extension*, seeing as it is applied so broadly as to be attributed to time as well, is the magnitude of the continuous. *Magnitude* is the multiplicity of parts" (Late 1671, "Specimen Demonstrationum de Natura Rerum Corporearum ex Phaenomenis" ("On the Nature of Corporeal Things: A Specimen of Demonstration from the Phenomenon"); A VI, 2, 306; LLC, p. 345). This text was written by Leibniz in order to demonstrate the insights appearing in the *TMA* (Arthur: LLC's 'Introduction', pp. 430-431, n. 31) and his commitment to the conceptual stance of the *TMA* is explict in what he later writes. Nonetheless, it is worth noting that in late 1671, Leibniz is no longer trying to claim that an unextended point is indivisible (as he claimed in the *TMA*). This change also appears in a letter to Arnauld in November 1671 and leaves the definition of a point as having magnitude but lacking extension as is.

25 "I once used to define magnitude as the number of parts, but later I considered that to be worthless, unless it is established that the parts are equal to each other, or of given ratio" (Early 1676, "De magnitude" ("On Magnitude"); A VI, 3, 482, quoted from *G. W. Leibniz. De Summa Rerum: Metaphysical Papers 1675-1676*, edited by G. H. R. Parkinson, New Haven – London 1992, p. 37).

as an elementary component of a continuum, the ratio $1 : \infty$ is preserved and therefore the division of the point will not change its extension but only the quantity of its internal parts. From the perspective of its extension, the point will always remain indivisible and therefore equal to all other points. However, when Leibniz speaks of one point being larger or smaller than another, he is saying that the quantity of their parts differs.

In Euclidean terms, the addition of a point to a line does not make the line any longer and therefore the attempt to compose a line using an infinity of points leads to a paradox in which the whole equals its part. In contrast, Leibniz in the *TMA* claims that there is a certain aspect in which the addition of a point *does* in fact increase the magnitude of a line. The extension of a line does not increase when an unextended point is added to it since the infinite ratio between them remains fixed whatever the case. But the addition of a point does in fact increase the number of points that make up the line:

"An *arc* smaller than any that can be given is still greater than its chord, although this is also smaller that can be expressed, i. e. consists in a point, but that being so, you will say, an *infinitangular polygon* will not be equal to a circle: I reply, it is not of an equal magnitude, even if it be of an equal extension: for the difference is smaller that can be expressed by any number"[26].

The distinction between extension and magnitude in the *TMA* is essentially that between continuous magnitude and fragmented magnitude based on actual parts and it allows Leibniz to make an interesting distinction between an infinite polygon and a circle. The difference between a circle and an infinite polygon is similar to that between an infinitesimal arc and an infinitesimal segment and is manifested only in the number of parts that make up their magnitudes. In an infinitesimal arc there is at least one part more than in an infinitesimal straight line, despite the fact that in the *TMA* both of them are considered to be indivisible and unextended points. The difference between extension and magnitude makes it possible to redefine the point and the composition of a continuum through a 'weak' infinite ratio ($1 : \infty$) instead of a 'strong' infinite ratio ($1 : 0$) and to evade the paradoxes of composing the continuum out of an infinite number of zeros and of equalizing the whole to its part[27]. This difference – between extension and magnitude – leads Leibniz to perceive the infinite polygon as identical but different from a circle and the line as identical but different from a curve. In the *TMA*, Leibniz is prepared to accept the situation in

26 *TMA* § 18; A VI, 2, 266; LLC, p. 342.

27 As explained by Bassler, it appears that the *TMA*'s weak spot is that Leibniz does not really justify this distinction between the two types of infinity (cf. O. B. Bassler: "The Leibnizian Continuum in 1671", in: *Studia Leibnitiana* XXX (1998), p. 6) and as a result the distinction between magnitude and extension is not completely clear. As a result, Leibniz eventually abandons this viewpoint and moves on to a theory that does not include points at all.

which a whole is equal to its part since different magnitudes (defined by their different number of actual parts) can be equal with respect to their extension[28].

When, in the winter of 1672, Leibniz rejects the idea that a continuum is made up of points, he is attempting to break free of the definitions that made it possible to create the *TMA* in the winter of 1670. It is impossible to claim that Leibniz could have resolved the Diagonal Paradox by distinguishing between the quantity of members in a set and its total magnitude (and thus allow for points to make up a continuum) since that is exactly the distinction he was trying to avoid. What is his justification for doing so? Why does Leibniz conclude in 1670 that in order for points to make up a continuum they must be defined as *partes extra partes* using the distinction between continuous extension and magnitude based on parts while in the winter of 1672 Leibniz uses that exact claim in order to avoid using this distinction and to completely reject the idea that a continuum is made up of points? Furthermore, in late 1672, Leibniz not only rejects the possibility of an unextended point as a component of the continuum, but also decisively concludes that a maximum is not possible either.

"*There is no maximum in things, or what is the same thing, the infinite number of all unities is not one whole, but is comparable to nothing*. For if the infinite number of all unities, or what is the same thing, the infinite number of all numbers, is a whole, it will follow that one of its parts is equal to it; which is absurd [...] We therefore hold that two things are excluded from the realm of intelligibles: minimum, and maximum; the indivisible, or what is entirely *one*, and *everything*; what lack parts, and what cannot be part of another"[29].

As mentioned above, Leibniz in the *TMA* switches to characterizing the ratio between a continuum and its components as the ratio between ∞ and 1 rather than that between 1 and 0. This implies that there is a difference between a continuum which is considered to be an infinite whole (1) and a continuum

28 It is known that in the 1680s, Leibniz defined the relationship between a subject and its predicates in an intensive sense. Nonetheless, in the *TMA* Leibniz still holds a mechanistic view as a result of the influence of Hobbes and Gassendi and therefore this intensive aspect was absent. For this reason, the relation in the *TMA* between the parts (which constitute the magnitude) and the whole (represented by the extension) is closer to the extensive sense in which Cantor perceived the relation between a class and its members. It cannot be said that this is a real bringing together of the two positions but rather is only a narrowing of the conceptual gap between them since it is quite clear that Leibniz did not distinguish between these two ways of defining a series of terms (L. Couturat: *La logique de Leibniz*, Paris 1901 (reprinted Hildesheim 1969), pp. 431-441). As shown by Nachtomy, Russell formulated his famous paradox of set theory in intensive terms (perhaps having been influenced by Leibniz's writings) and only afterward made the appropriate adjustments in extensive terms in order to make the paradox relevant to Frege's logic (O. Nachtomy: "Leibniz and Russell: the Number of All Numbers and the Set of All Sets", in: P. Phemister and S. Brown, *Leibniz and the English-Speaking World*, Dordrecht 2007, pp. 207-218, p. 211). Therefore, in my opinion, the differences between the two viewpoints on the relationship between a whole and its parts are not critical here.

29 Nov. 1672-Jan 1673, "De minimo et maximo. De corporibus et mentibus" ("On Minimum and Maximum. On Bodies and Mind"); A VI, 3, 98; LLC, p. 13.

that is considered to be simply an infinite (∞). This difference would seem to indicate that Leibniz already recognized in the *TMA* that an infinite whole is not possible. However, the opening of the *TMA* indicates a declared commitment to the existence of an actual infinite whole[30]. It seems that Leibniz leaves the extent of his commitment to the existence of an infinite whole unclear because, as mentioned above, he was not yet convinced that the existence of a maximum or an infinite number is to be rejected. However, the connection between an infinite and a whole is entirely rejected by Leibniz in the winter of 1672 using the same argument as in the *TMA*. What causes Leibniz to change his mind about the status of the infinite and the infinitesimal?

In late 1672, Leibniz became convinced that an infinite number is contradictory and therefore impossible since the whole is then equal to its part. In the *TMA*, there is still the possibility that the whole will be equal to its part and thus Leibniz still supported the idea that there is an actual infinite number of parts in a continuum. However, on his arrival in Paris, Leibniz came to the conclusion that the Euclidean axiom concerning the whole being larger than its part can be proven using his Principle of Contradiction. This means that from this point on, Leibniz begins to relate to the whole being larger than its part as a proven principle whose truth is eternal, independent of context and not subject to doubt. For this reason, he is forced to sweepingly deny the possibility of an infinite number and even to reject the distinction between extension and magnitude which he utilized in the *TMA* to claim that an infinite polygon and a circle are different but equal. If there is a mistake in Leibniz's arguments, it is the status of the necessity of eternal truth that Leibniz attributes to his Principle of Contradiction through which he proved the Euclidean part-whole axiom. Indeed, the fact that there exist two standards for the comparison of magnitudes is correct though in the historical context of Leibniz's usage of the Diagonal Paradox it is insufficient for him to allow for a continuous line to be made up of points.

As I have tried to show here, I think we can defend a denial of the composition of the continuum from points but not in the way that Leibniz eventually did. My defense focuses on the philosophical context in which Leibniz excluded the point as a component of the continuum. But in the end this defense converges to Leibniz's conviction that the Principle of Contradiction is not contingent. In other words, Leibniz is completely justified in rejecting the notion of infinite and infinitesimal magnitudes if indeed his Principle of Contradiction is necessary. But it is not, as shown by quantum mechanics logic, in a state of superposition which implies A and ~A simultaneously. It is easier to contradict a necessity than to negate a possibility. Leibniz defines his statements on the infinite as a necessary truth, maintaining that an infinite number is a contradictory notion and as such it is impossible; thus an opposite possibility

30 "There are actually parts in the continuum, contrary to what the most acute Thomas White believes, and *these are actually infinite*, for Descartes's 'indefinite' is not in the thing, but the thinker" (*TMA* § 1-2; A VI, 2, 264; LLC, p. 339).

like Cantor's version of the infinite can negate the validity of Leibniz's argument. The same is true with respect to some of Leibniz's commentators. The critique of Leibniz's usage of the Diagonal Paradox does not seem to take into account Leibniz's philosophical assumptions and refers to the possibility of an infinite whole or infinitesimal point as an unquestioned and a-historical issue. This mathematical critique of Leibniz can also be deflected by examining the context in which Leibniz makes his claims. It is wrong to examine a statement, even a mathematical one, out of context. But this means that the historical context in which Leibniz proves his view on the infinite, the infinitesimal and the continuum creates difficulties for his absolute justification. Unfortunately, this kind of effort to understand Leibniz's position cuts down the very branch on which Leibniz was sitting.

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